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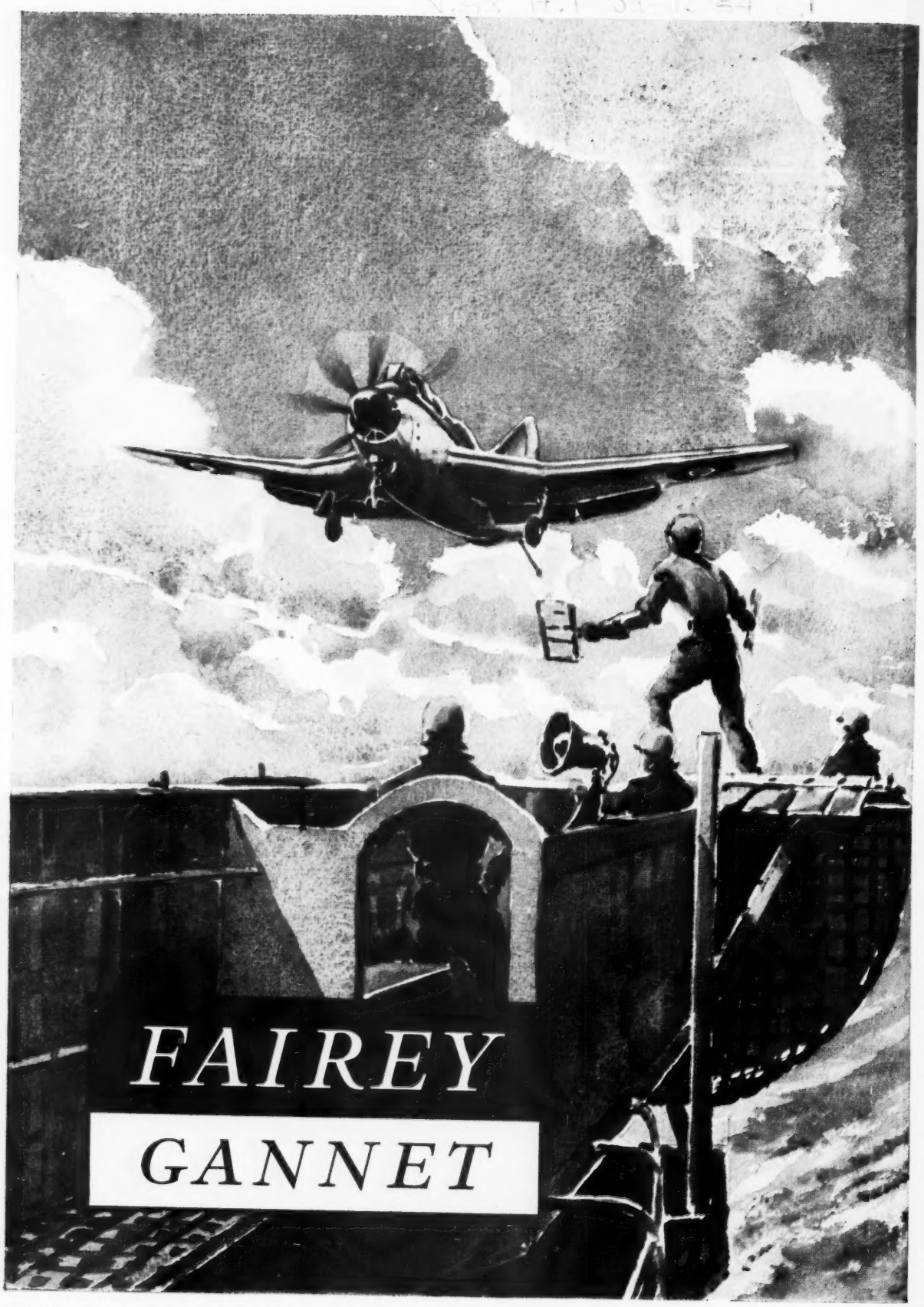

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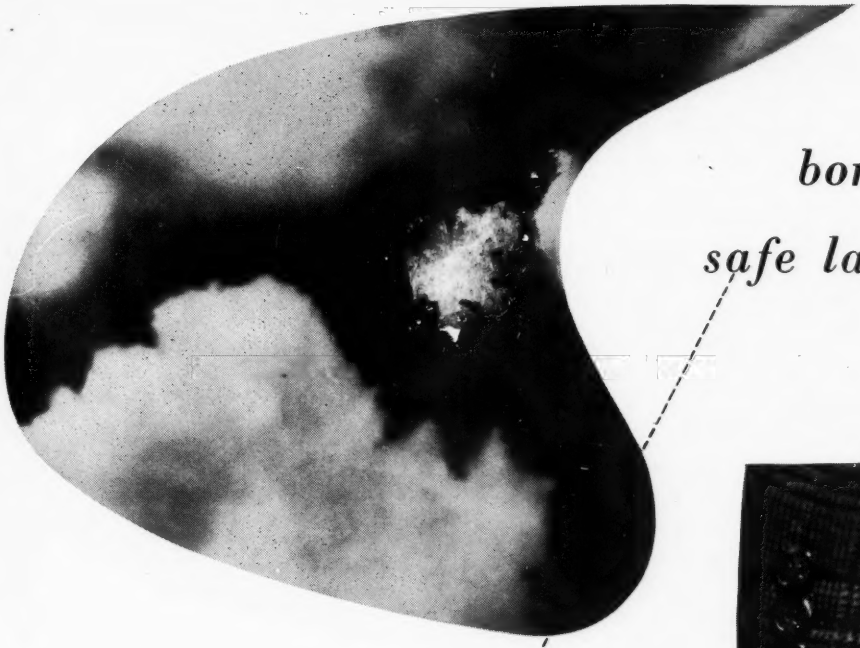

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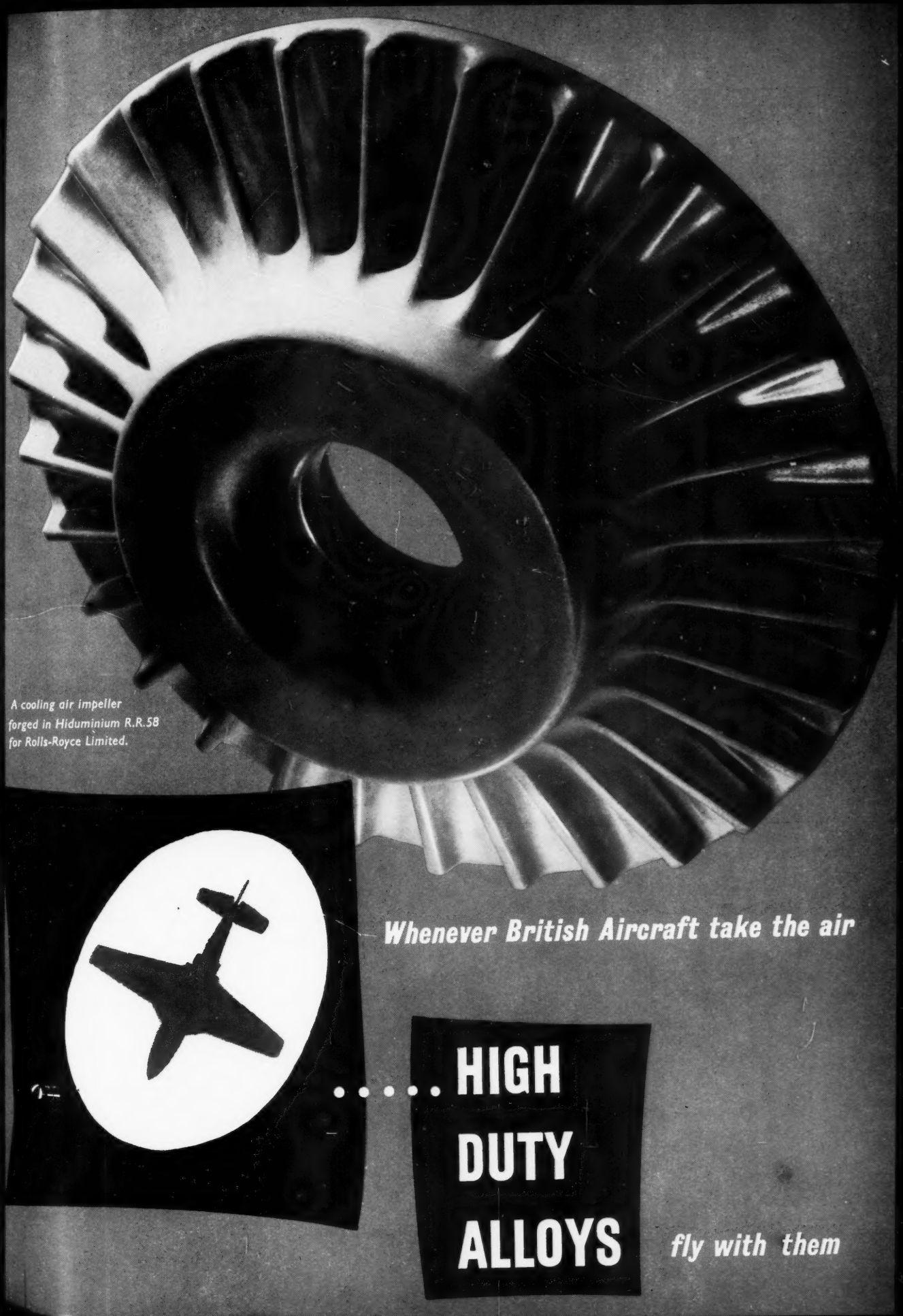
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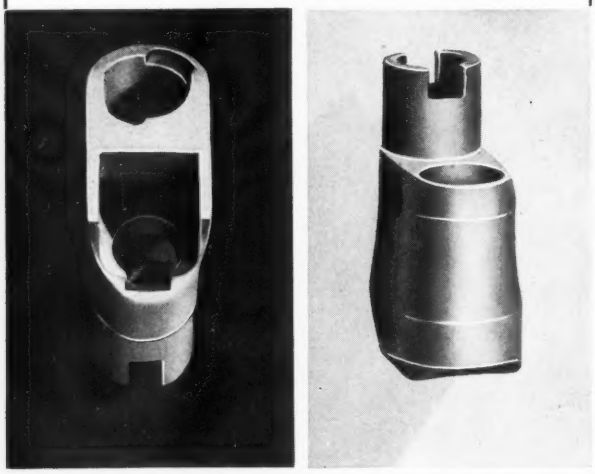
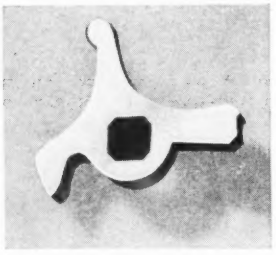
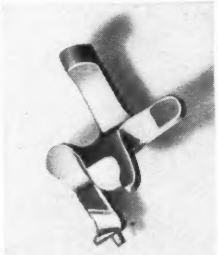


Two views of a rifle component in a Silicon Manganese alloy steel.



Another rifle component in 0.4% carbon steel.



Two views of a tabulator dog-lever in 3% Ni case-hardening steel.

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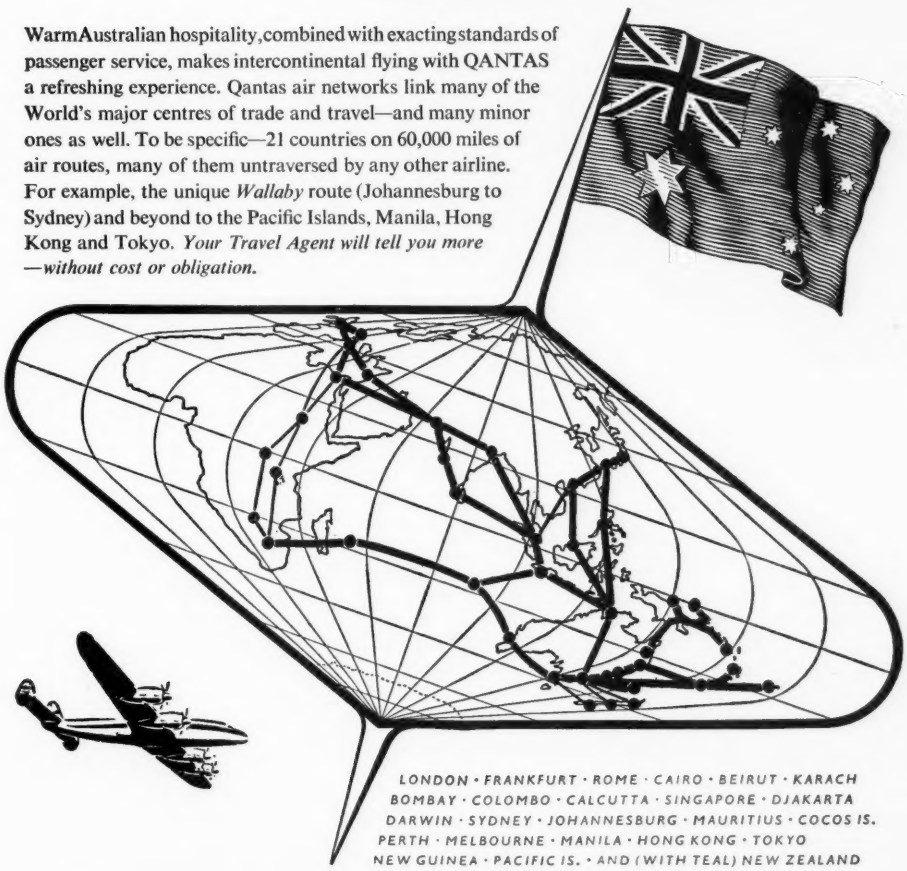

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VOLUME 58 JANUARY 1954 NUMBER 517

From the President

THE year 1953 has been one of great significance for the Society, as on 17th December we celebrated the 50th Anniversary of the first flights of a machine which carried its own means of propulsion. It was flown by Orville and Wilbur Wright, whom the Society honoured by the presentation, for the first time in its history, of its Gold Medal.

I write this on 18th December, with a feeling that our efforts to honour their memory, at our Dinner, in which our friends of the Royal Aero Club joined, was worthy of the historical occasion. No one who was present will forget the eloquent tribute of Mr. Alan Lennox Boyd, the Minister of Transport and Civil Aviation, or the moving words of Lord Brabazon.

The burden on the Council and Staff has been heavy, but they have borne it gladly because they believed all Members of the Society would support them in their efforts to ensure that an occasion unique in the history of flying was worthily honoured. As President, my task has been lightened by their unselfish work and I take this opportunity of thanking them.

I have had many personal expressions of friendship and support from Members all over the country. Without this my task would have been very difficult. With it, and with the loyal support of the Council and Staff, I look forward with pleasure to the remaining months of my time of office.

I now have an announcement to make in connection with the Nash Collection of Historical Aircraft.

The Council learnt some months ago that Mr. R. J. G. Nash had decided that he would soon be obliged to dispose of his collection of historic aircraft. Mr. Nash had long hoped that these might form the nucleus of a National Air Collection. The Council was favourably disposed to the suggestion that there should be such a collection, but, particularly in view of the problem of storing full sized aircraft, had not yet been in a position to put forward a considered proposal with the object of securing public support.

In November, however, an offer to buy Mr. Nash's aircraft was received from the United States, and the fate of the collection became a matter of urgency. Mr. Nash was anxious that it should not leave this country. The Council considered the situation at its meeting on 26th November and unanimously decided that an attempt should be made to buy the collection. Mr. Masefield, with his Technical Assistant Mr. Brooks, negotiated with Mr. Nash and a price, within the sum authorised by the Council, was agreed on 10th December. I authorised the completion of the purchase on behalf of the Society.

A further announcement will be made as soon as the Council has considered how best to deal with the future of these aircraft.

They are as follows:—

Bleriot X1A (1910), Bleriot XXVII (1911), Caudron G.3 (1912), Maurice Farman F.40 (1913), Avro 504 (1914), Sopwith Camel (1917), Fokker D.VII (1918), S.E.5A (1918).

I believe that all Members will approve the Council's decision to take this step, and particularly at this time.

I send warmest greetings for the New Year to all Members.

President

NOTICES

An annual sum of £250 is available for premium awards, usually 15 guineas each, for papers published in the Journal. Members and non-members of the Society are invited to submit papers on any aspect of aeronautics

Nominations for Fellowship of the Society

The attention of Members is drawn to the following extract from the Society's By-Laws:—

"Fellows shall comprise every person who on the 30th day of January 1950 was on the Register as a Fellow of the Society; and every person thereafter awarded the honour of Fellowship.

"Elections to Fellowship will be made annually by the Council and will be announced at the Annual General Meeting of the Society. Nominations will be initiated by the members of the Council or by any four Fellows of the Society. It is the duty of the Council to see that the honour is awarded only to persons who have attained a considerable degree of technical eminence in the profession of aeronautics."

From the above it will be seen that any four Fellows may nominate a suitable candidate for Fellowship.

Graduates' and Students' Section—Imperial College

A visit to the Imperial College of Science and Technology has been arranged for Wednesday 27th January 1954, at 6.30 p.m. Visitors will be able to see the wind tunnels, structural test rigs and so on. Supper will be available at the College at Moderate charges. Applications, stating whether dining or not, should be sent to the Hon. Visits Secretary, D. W. J. Richards, 18 Geneva Road, Kingston-on-Thames, Surrey.

Annual Subscriptions

Members are reminded that their annual subscriptions became due on 1st January 1954. The rates are:—

	Home	Abroad
	£ s. d.	£ s. d.
Fellows	5 5 0	4 4 0
Associate Fellows	4 4 0	3 3 0
*Associates	3 3 0	3 3 0
Graduates (aged under 26)	2 2 0	2 2 0
Graduates (aged 26 and over)	2 12 6	2 12 6
Students (aged under 21)	1 1 0	1 1 0
Students (aged 21 and over	1 11 6	1 11 6
Companions	3 3 0	3 3 0
Founder Members	2 2 0	2 2 0

* Any Associate elected before 1st October 1947 may, if he wishes, elect not to receive the Journal, and in this case his subscription will be reduced by £1 1s. 0d. to £2 2s. 0d.

It will avoid delay and confusion if members, when sending remittances for subscriptions, will state their names clearly and give their addresses and grades of membership. Remittances should be made payable to the Royal Aeronautical Society.

Changes of Address

To assist in keeping the records of members correct and up to date the Secretary will be glad if all members will notify him as soon as possible of changes of address.

When notifying changes please give the following particulars:—

Name (in block letters). New address (in block letters).

Grade of membership. Old address.

Changes of address must be received *before the 15th of the month in order to be effective for the* Journal for the following month.

News of Members

Captain T. H. Farnsworth (*Associate Fellow*) has been awarded the British Airline Pilots' Association first Gold Medal.

Major P. L. Teed (*Fellow*) has been elected Vice-President of the Institute of Metals.

Members' New Appointments

J. S. Butcher (*Associate Fellow*) has been appointed Chief Draughtsman of Fields Aircraft Services Ltd., Nottingham.

A. F. Duncan (*Graduate*) has taken up a position as a Propulsion Engineer in the Technical Design Section of the Fort Worth Division of the Consolidated Vultee Aircraft Corporation, Texas.

R. W. Grandorge (*Associate Fellow*) has been appointed Deputy Resident Technical Officer at Armstrong Siddeley Motors Ltd.

H. C. Harrison (*Associate Fellow*), Chief Executive (Engineering) of Fight Refuelling Ltd., has been appointed a Director of the Company.

Flt. Lt. W. Heaton, M.B.E. (*Associate Fellow*) has taken up an appointment within the Directorate of Servicing Research and Development at the Ministry of Supply.

G. S. Henderson (*Associate Fellow*), formerly Technical Assistant (Development) at Rotol Ltd., Gloucester, has been appointed Technical Assistant (Development) at D. Napier & Son Ltd., Luton.

D. M. Heughan (*Graduate*) has been appointed a Senior Technical Assistant in the Aerodynamics Department of the de Havilland Aircraft Company.

L. E. Leavy (*Associate Fellow*) is now in charge of the wind tunnels at A. V. Roe & Co. Ltd., Woodford, Cheshire.

C. Morley (*Associate Fellow*), formerly of the Ministry of Supply T.R.E. Aircraft Department, has recently taken up appointment as Research Engineer with the Lockheed Aircraft Corporation, Burbank, California.

E. F. Needham (*Associate Fellow*) has been appointed Chief Stressman of the Airspeed Division of the de Havilland Aircraft Company.

A. H. Old (*Associate Fellow*) has taken up an appointment as Head of the Mechanical Engineering Department at Rugby College of Technology and Arts.

R. K. Page (*Associate Fellow*) has taken up the post of Chief Technician of Folland Aircraft Ltd.

F. H. Pollicutt (*Fellow*) has been appointed Chief Designer to Percival Aircraft Ltd. He was formerly Chief Designer, Folland Aircraft Ltd.

A. W. Rogers (*Associate Fellow*) has been appointed an Engineer II in the Directorate of Aircraft Production Development, Ministry of Supply (Air).

D. J. Tyler (*Graduate*), formerly technical assistant (development) with D. Napier & Son Ltd., is shortly to take up an appointment as Experimental Engineer with Megator Pumps and Compressors Ltd.

G. S. Webley (*Associate*) has taken up the appointment of Director and General Manager of Cape Engineering Company Ltd., Warwick.

J. A. C. Williams (*Associate Fellow*) has taken up a Research Fellowship at the College of Aeronautics, Cranfield.

J. B. Witchell (*Associate Fellow*) has taken up an appointment as Defence Research Technical Officer at the Canadian Armament Research and Development Establishment, Valcartier, Quebec.

Diary

LONDON

January 7th 1954
LECTURE TO YOUNG PEOPLE. The Fun of Finding Out in Flying. Sir Leonard Bairstow, C.B.E., F.R.S., Hon.F.R.Ae.S. At the Institution of Mechanical Engineers, Storey's Gate, S.W.1. 3 p.m. (Tea will be served after the lecture.)

January 21st
MAIN LECTURE AT THE SOUTHAMPTON BRANCH: First Mitchell Memorial Lecture. J. Smith, C.B.E. Institute of Education, University of Southampton. 7 p.m.

January 26th
SECTION LECTURE. X-Ray Metallography. Dr. G. L. J. Bailey. 4 Hamilton Place, W.1. 7 p.m.

February 2nd
GRADUATES' AND STUDENTS' SECTION. The Structural Analysis of Swept-back Wings. D. Howe. **Joint Meeting with the London Graduates' Section of the Institution of Mechanical Engineers.** At the Institution of Mechanical Engineers, Storey's Gate. At 6.30 p.m. (Tea at 6 p.m.)

February 4th
SECTION LECTURE. Air Intake Efficiency. F. B. Greatrex. 4 Hamilton Place, W.1. 7 p.m.

February 11th
MAIN LECTURE. Recent Developments in the Structural Approach to Aeroelastic Problems. Dr. D. Williams. At the Institution of Mechanical Engineers, Storey's Gate, S.W.1 6 p.m. (Tea at 5.30 p.m.)

February 18th
SECTION LECTURE. Helicopter Research. (Joint Lecture with Helicopter Associaion). F. O'Hara. 4 Hamilton Place, W.1. 7 p.m.

February 24th
GRADUATES' AND STUDENTS' SECTION. Aircraft Plastics. Lorne Welch. 4 Hamilton Place, W.1. 7.30 p.m.

March 2nd
SECTION LECTURE. Metal Sandwich Construction. F. Tyson. 4 Hamilton Place, W.1. 7 p.m.

BRANCHES

January 6th
Belfast.—Second Joint Meeting with Queens University, Belfast, Film Society. Films include: (1) R.Ae.S. Film Commemorating 50 Years of Powered Flight; (2) Travel Royal. Whitla Hall, Queen's University, Belfast. 7 p.m.
Brough.—New Developments in Aircraft Production Engineering. Professor J. V. Connolly. Electricity Buildings, Hull. 7.30 p.m.
Chester.—Guided Weapons. D. J. Lyons. Grosvenor Hotel, Chester. 7.30 p.m.
Luton.—Wilbur Wright Centenary Film. George Hotel. 7.30 p.m.
Southampton.—Annual General Meeting, followed by Films. Institute of Education, The University. 7 p.m.

January 11th
Birmingham.—Joint Meeting with the Institute of Rubber. James Watt Institute, Great Charles Street, Birmingham. 7.30 p.m.
Halton.—The Propeller Gas Turbine Engine. A. C. Clinton. Branch Hut, R.A.F. Station, Halton. 6.45 p.m.

January 13th
Hatfield.—Stability and Control in Aircraft Design. J. C. Wimpenny. de Havilland Restaurant, Hatfield. 6.15 p.m.
Weybridge.—Sailplane Design. K. G. Wilkinson. Vickers-Armstrongs Ltd., Weybridge Works. 6 p.m.

January 14th
Isle of Wight.—Large Press Forgings. E. T. Stewart-Jones. Club House, Saunders-Roe Sports and Social Club, Church Path, E. Cowes. 6.30 p.m.

January 15th
Glasgow.—The Pilot's Point of View. Officers of the University Air Squadron. Royal Technical College. 7.30 p.m.

January 18th
Halton.—Branch Night. Branch Hut, R.A.F. Station, Halton. 6.45 p.m.

January 19th
Belfast.—Flight Testing Problems. D. N. Scard. Kerr Room, Kensington Hotel, Belfast. 7 p.m.

January 20th
Bristol.—Design Problems of the Transport Helicopter. R. Hafner. Conference Room, Bristol Aeroplane Co. Ltd. 6 p.m.
Coventry.—Recent Developments in Gliding. A. H. Yates. Wine Lodge, Coventry. 7.30 p.m.
Leicester.—Annual General Meeting and Film Show. 1 Newarke Street, Leicester. 7.15 p.m.
Manchester.—Lecture. Reynolds Hall, College of Technology. 7.30 p.m.
Preston.—Stability and Control of Delta Wing Aircraft. Wg. Cdr. R. J. Falk. Queen's Hotel, Lytham. 7.30 p.m.

January 21st
Southampton.—MAIN LECTURE. First Mitchell Memorial Lecture. J. Smith. Institute of Education, The University. 7 p.m.

January 25th
Halton.—Junior Members' Night. Branch Hut, R.A.F. Station, Halton. 6.45 p.m.

January 27th
Weybridge.—Space Flight. J. Humphries. Vickers-Armstrongs Ltd., Weybridge Works. 6 p.m.

February 1st
Derby.—Aeronautical and Industrial Application of Small Gas Turbines. F. R. Bell. Rolls-Royce Welfare Hall, Nightingale Road, Derby. 6.15 p.m.
Halton.—Films. Branch Hut, R.A.F. Station, Halton. 6.45 p.m.

February 2nd
Bristol.—Operating Problems of the Transport Helicopter. P. G. Masefield. Conference Room, Bristol Aeroplane Co. Ltd. 6 p.m.

February 3rd
Chester.—Pioneer Flying. H. Shaw. (Joint Meeting with the Stanlow Branch of the Institute of Petroleum). Grosvenor Hotel, Chester. 7.30 p.m.
Brough—The Design, Construction and Flight Testing of the Universal Freighter. C. W. Prower, W. A. Hargreaves and G. R. Parker. Lecture Hall, Electricity Buildings, Ferensway, Hull. 7.30 p.m.

February 4th
Isle of Wight.—The Development of Propellers. H. R. Leather and P. Brett. Clubhouse, Saunders-Roe Sports and Social Club, Church Path, E. Cowes. 6.30 p.m.

February 6th
Birmingham.—Annual Dinner. White Horse Restaurant. 7.30 p.m.

February 8th
Halton.—Flight Simulators. G. H. Hellings. Branch Hut, R.A.F. Station, Halton. 6.45 p.m.

February 9th
Belfast.—Engineering Problems of Airline Operations. B. S. Shenstone. Kerr Room, Kensington Hotel, Belfast. 7 p.m.

February 10th
Bristol.—Joint Meeting with the Institute of Production Engineers. Lecturers to be announced later.
Weybridge.—Navigational Aids. Wing Commander G. N. Macintosh, O.B.E. Vickers-Armstrongs Ltd., Weybridge Works. 6 p.m.

February 12th

Glasgow.—Film Evening. (The R.Ae.S.—Shell Film, "Powered Flight—The Story of the Century" will be among those shown.) Royal Technical College. 7.30 p.m.

February 15th

Halton.—Junior Members' Night. Branch Hut, R.A.F. Station, Halton. 6.45 p.m.

February 17th

Coventry.—The Development of the Bristol Proteus Turbine Engine. Dr. S. G. Hooker. Wine Lodge, Coventry. 7.30 p.m.

Manchester.—The Rocket Power Plant. S. Allen. Reynolds Hall, College of Technology, Manchester. 7.30 p.m.

February 18h

Leicester.—Present Trends in Aircraft Materials and Construction. W. A. Baker. Leicester College of Technology. 7.15 p.m.

February 23rd

Belfast.—Engineering Problems of Airline Operations. B. S. Shenstone. Kerr Room, Kensington Hotel, Belfast. 7 p.m.

February 24th

Belfast.—The Short Analogue Computor. E. Lloyd-Thomas. Canteen, Short Brothers and Harland Ltd., Castlereagh, Belfast. 7 p.m.

Weybridge.—The Shape of Wings to Come. D. Keith-Lucas. Vickers-Armstrongs Ltd., Weybridge Works. 6 p.m.

February 25th

Isle of Wight.—Accident Investigation. Air Commodore Sir Vernon Brown. Clubhouse, Saunders-Roe Sports and Social Club, Church Path, E. Cowes. 6.30 p.m.

February 26th

Birmingham.—The Development of the Vickers Viscount. A. Greenwood. Birmingham Chamber of Commerce. 7.30 p.m.

March 1st

Halton.—Films. Branch Hut, R.A.F. Station, Halton. 6.45 p.m.

Elections

The following is a list of new members and transfers of membership of the Society:—

Associate Fellows

Robert Morrison Aldwinckle
John Nelson Boothman
Eric Raymond Bulmer (*from Associate*)
Brian Edwards (*from Graduate*)
John Outram Emmerson (*from Graduate*)
Thomas Harry Farnsworth (*from Associate*)
Oliver Logie Lloyd Fitzwilliams
Gabriel Maria Giannini
Nigel Gregory (*from Graduate*)
David Harris (*from Graduate*)
James Charles Hateley (*from Graduate*)
Hall Livingstone Hibbard
Gordon Thomas Hudson (*from Graduate*)
Albert Edward Hurren (*from Graduate*)
Simon Kugler (*from Graduate*)
Daniel Liptrot
Raymond L. Nivet
Lionel Rossiter Pool (*from Graduate*)
Alan William Ratcliff (*from Graduate*)
Frederick Robert Robinson (*from Graduate*)
Arnold Rose (*from Graduate*)
Robert James Ross (*from Graduate*)
Peter Latham Sutcliffe (*ex Graduate*)
Denis Campbell Tufnell
Philip Henry Vernon Vine
Michael Thomas Witts (*from Graduate*)

Associates

Frederick Ainslie Eskdaill Allan
Charles William John Allen
Kenneth Gordon Archer
Geoffrey Broughton Foort
Edward Duncan Jones
Harold Jones
John Thomas Roberts
Maurice Dudley Smith
Reginald Donald Whitmore
Frank William Willcox

Graduates

Arthur Akers
Michael James Clarke (*from Student*)
Gerald Kenneth Hunt (*from Student*)
Alun David Jones
Laurence Henry Kimberley
Charles Eric Bruce McConachie
Walter John Paul
David William Rowe (*from Student*)
Rakhal Das Saha (*from Student*)
Stanley Sherman
Gordon Louis Taylor
Robert Ernest Yapp

Students

Andrew Achilles Askotis
Clive Athelstan Bainbridge
Martyn Joseph Ernest Callow
George Alan Failey
Theodore Alexander Ginsburg
Malcolm Lloyd Hayward
Alister Denny Raby
Anthony Goodwin Roberts

Companions

Rollin Hamilton Bates (*from Graduate*)
George Guthrie Petrie Thorburn

Acknowledgments

The Council acknowledge with grateful thanks the gift of a handsomely bound Catalogue of the Aero and Motor Boat Exhibition, Olympia, 1909, from Wing Commander E. D. Hamilton Ross, and the return of back numbers of the Journal from V. H. Wickham, Esq., *Associate Fellow*.

Journal Binding

Self-Binder Cases

Self-Binder cases of the "Easibind' type are available from the offices of the Society. These binders are for members who do not have their Journals permanently bound, or who wish to keep their Journals together during the year for binding later.

These cases will hold 12 Journals which are kept in place by means of flexible steel wires. Journals can be inserted or withdrawn easily without damage, so preserving the contents for permanent binding later. The Journals will open flat at any page.

The binder is strongly made in durable dark blue leather cloth on stiff board covers and has gold lettering on the spine. The year is not blocked on the spine but there is a panel on which members who wish to use the binder as a permanent case can put the date.

The cost is 11s 6d. each including postage and packing for either the size to fit 1952 and previous Journals, or for the size to fit the Journal from January 1953, which has been increased in size. Orders and remittances should be sent direct to the Secretary at the Offices of the Society and it is *important to state whether the old size or new size is required.*

Permanent Binding

There is no increase in the price of permanent binding of Journals. The prices are:—

1953 Volume (including packing and postage) 16s. 0d.
Previous Volumes (including packing and postage) 18s. 0d.

Journals, *with a note of the name and address of the sender*, should be sent direct to The Lewes Press, Friars Walk, Lewes, Sussex, and the remittance to the Secretary at the Offices of the Society.

Journal of The Royal Aeronautical Society

WITH WHICH IS INCORPORATED THE INSTITUTION OF AERONAUTICAL ENGINEERS

VOLUME 58 JANUARY 1954 NUMBER 517

The Forty-First Wilbur Wright Memorial Lecture

THE FORTY-FIRST Wilbur Wright Memorial Lecture, on "Buckling and Stability," was given by Professor N. J. Hoff on 14th September 1953 at the Royal Institution, 21 Albemarle Street, London, W.1, before a large and distinguished audience, which included many Americans attending the Fourth Anglo-American Aeronautical Conference. Sir William Farren, President of the Society, presided at the meeting.

Medals and awards were presented by the President before the lecture.

SIR WILLIAM FARREN said that in paying their yearly tribute to the memory of the man who made the first mechanically propelled flight they honoured also his brother, Orville Wright, who had been known to many of them. The fiftieth anniversary of that event fell on 17th December 1953, and it was the Society's intention to celebrate it in an appropriate manner.

Since 1929 it had been the Society's custom to invite, every other year, someone from the United States of America to deliver the Wilbur Wright Lecture. Even before that date there had been four occasions on which Americans gave the lecture, the first being Dr. W. F. Durand in 1918. The lecture this year had been arranged to coincide with the opening of the Fourth Anglo-American Aeronautical Conference and he welcomed many members of the Institute of the Aeronautical Sciences. The date happened to fall in a week which was possibly one of the most significant in the whole history of flight, when we in this country, with good reason, remembered the Battle of Britain.

It was the custom on this occasion to present the senior honours of the Society to those who had been chosen to receive them. The highest tribute they could pay to anyone was to offer him the Honorary Fellowship of the Society. This year the Council had selected three names which, he felt, would be universally approved—Sir Geoffrey de Havilland, Sir Arthur Gouge, and Lord Hives.

In presenting the scrolls, the President said that Sir Geoffrey's achievements were equalled only by his modesty. He had never heard him make a speech! To Sir Arthur, a former President, he expressed their admiration for his lifetime of devotion to the development of the large flying boat. Lord Hives was unable to be present, but he hoped to hand him his scroll in that week. There were few men more worthy of such recognition in "Battle of Britain" week than the man who made the Merlin engine.

Before passing to the Society's Medallists, Sir William said he felt he should pay a tribute to the memory of one of the greatest names in the history of aviation, that of a man who had honoured the Society by accepting the Honorary Fellowship twenty-six years ago—Professor Prandtl. Only three others had received all of the Society's three highest honours—the Honorary Fellowship, the Gold Medal, and the invitation to deliver the Wilbur Wright Lecture. His work had had a profound influence in many fields of science and engineering. It had produced a complete revolution in aerodynamics. Prandtl was a man of great simplicity and a delightful friend. They would be glad to remember that he was able to continue working almost to his death a few days ago at the age of 78.

Blackstone Studios, New York

PROFESSOR N. J. HOFF, Forty-first Wilbur Wright Memorial Lecturer, 1953.

The Society's Gold Medal—The award of the Society's Gold Medal to Mr. E. F. Relf was a recognition of a remarkable achievement. After 21 years as Head of the Department of Aeronautics at the National Physical Laboratory, Mr. Relf had devoted five years to setting in motion the great College of Aeronautics at Cranfield — a service to aviation and to the country which the Society acknowledged with gratitude.

The Society's Silver Medal—To Mr. H. Grinsted, the President presented the Society's Silver Medal in recognition of a lifetime of work of the highest value to aeronautical engineering, in particular at the Ministries of Aircraft Production and Supply. There were few of them who did not have mixed feelings about such places. But if they were necessary, they must have good engineers in them, and no one had been higher in their regard and respect as an engineer, than Mr. Grinsted.

The Society's Bronze Medal—Presenting the Society's Bronze Medal to Mr. L. Boddington, Sir William said that his long record of achievement at the

SIR GEOFFREY DE HAVILLAND, Honorary Fellow, 1953.

MR. E. F. RELF receiving the Society's Gold Medal.

Royal Aircraft Establishment and elsewhere in aid of naval aviation was something of which he had intimate personal knowledge. In addition to his work on catapults and arrester gears, his name was associated with the "flexible" deck, and he shared the responsibility for the idea of the "angled" deck.

The British Gold Medal for Aeronautics—To Mr. R. E. Bishop, who had been awarded the British Gold Medal for Aeronautics, the President expressed their admiration for the achievements of the Comet, which was at that moment engaged in an enterprise which would add one more feather to the many in Mr. Bishop's cap.

The British Silver Medal—The British Silver Medal for Aeronautics was awarded to Mr. J. E. Gordon. The advance of plastics as structural materials in aviation was largely due to the activities of Mr. Gordon and his colleagues at the Royal Aircraft Establishment.

The Simms Gold Medal—The award of the Simms Gold Medal to Major P. L. Teed was a recognition of his paper "The Fatigue of Aircraft Materials with Special Reference to Micro-Structure." Fatigue of aircraft was now very much in everybody's mind and Major Teed was a magnificent exponent of its underlying principles.

The Wakefield Gold Medal—The Wakefield Gold Medal had been awarded to Mr. F. W. Meredith for his work on the design of automatic pilots and aircraft instruments. Mr. Meredith had been working on instruments for aviation for almost as long as he could remember, and yet his inventive capacity seemed to be entirely undimmed.

The George Taylor (of Australia) Gold Medal—The George Taylor (of Australia) Gold Medal had been awarded to Group Captain E. A. Whiteley, at present in Singapore, for his paper on "The Spacing of Aircraft under High Density Approach Conditions." Few subjects were of more concern to aircraft operators.

Introducing Professor N. J. Hoff, the Wilbur Wright Memorial Lecturer for 1953, the President said the Lecturer shared one distinction with a great friend and Honorary Fellow of the Society, whom they were delighted to see present; like Dr. von Kármán he was by birth a Hungarian. Professor Hoff went to America in 1939 and was now the Head of the Department of Aeronautical Engineering and Applied Mechanics at the Polytechnic Institute of Brooklyn, New York. He was well known to a wide circle in this country and he welcomed him on behalf of the Society.

MR. H. GRINSTED receiving the Society's Silver Medal.

MR. R. E. BISHOP receiving the British Gold Medal.

FORTY-FIRST WILBUR WRIGHT MEMORIAL LECTURE

Buckling and Stability

by

N. J. HOFF*

Introduction

Although columns built almost 5,000 years ago still stand in Egypt, the first theoretical calculation of the load-carrying capacity of columns was published by Euler only 209 years ago. The classical theory established in this publication is still the foundation of all stability analyses and is likely to remain so in the future. It was derived for perfectly elastic columns and for specific end conditions. These end conditions are not realised in the ordinary column test nor do they correspond exactly to the conditions prevailing in an aeroplane wing when it is subjected to a gust or to the shock of landing. They are not the same as the end conditions of a column in a building when it collapses during a tornado or an earthquake.

It is a fortunate coincidence from the standpoint of practical applications, but perhaps unfortunate for the development of a more accurate theory, that the column test carried out in a routine manner in an ordinary testing machine gives results in excellent agreement with the classical theory when the column is slender. The maximum load observed in the test is the same as the Euler load and it is maintained for a considerable length of time while the loading head is moving downward. But the situation is different with short columns whose buckling stress exceeds the elastic limit of the material. They reach a maximum value of the load after which the load drops gradually, or with a sudden snap, depending on the properties of the testing machine. The maximum value of the load is always considerably smaller than that predicted by the Eulerian theory; its calculation has been the subject of many excellent technical papers in the past sixty odd years.

There are even greater deviations from the behaviour predicted by the classical theory than that observed with the short column. The thin-walled circular cylindrical shell collapses catastrophically under a load which amounts to only one-third of the classical limit of stability; the simply supported flat plate can be subjected to loads exceeding the theoretical stability limit by a factor of ten or more if it is thin; in a very rapid test the ordinary column can support a multiple of the critical load; and an aluminium alloy column maintained at a uniform temperature of 500°F. will collapse, with a snap, under any compressive load, however small, if the test is continued long enough.

This short description shows how manifold column behaviour can be under various conditions of loading. Many features of the buckling process cannot be explained on the basis of the classical concept of stability. Other means must be found therefore to satisfy the needs of the practicing engineer, and particularly those of the aeroplane structures man who has to guarantee the safety of the structural elements of supersonic aircraft when they are subjected to high temperatures and rapidly varying loads.

Why is the structural engineer interested in stability? To him stability means that he can rely upon the equilibrium of the structure he designed; he need not be afraid that the structural element may collapse in consequence of the vibrations and the changes in loading that can be expected to occur in service. Equally as objectionable as complete collapse are deformations of a magnitude that interferes with the proper functioning of the structure. In the presence of non-linear effects stability should be established through a dynamic analysis of the loading process, or through a calculation of the motion following a disturbance of the original equilibrium.

The purpose of this paper is to present a detailed investigation of the behaviour of both slender and short columns in the ordinary column test and to correlate the buckling loads obtained in the test with the critical loads of the classical theory. In addition, the effects of rapid load application and of creep are discussed at some length. Experiments are described which were conducted to confirm some of the conclusions of the theory. The history of the development of the theory of stability is sketched and some of the significant features of the buckling of plates and shells are mentioned. It is suggested that a clear distinction be made between the critical load obtained from a stability analysis in which the load has a prescribed value, and the buckling load under which a compressed element collapses in service under varying loads or buckles in a loading test.

Notation

A	total cross-sectional area of column
a	non - dimensional total deflection of middle of column
a_i	non-dimensional deflection of middle of column in i^{th} mode
a_0	non-dimensional deflection of middle of column at the instant when creep begins
a_s	velocity of sound
a^*_1	non-dimensional displacement of middle of arch in first mode
a^*_2	non-dimensional displacement of middle of arch in second mode
b, b^*	non-dimensional moment at middle of column
C	spring constant

*This paper was prepared in connection with research contract N6-ONR-26308 sponsored by the Office of Naval Research of the U.S. Navy.

C', C^*, C_1 constants of integration
c loading head velocity
d differential sign
e non-dimensional initial deviation of middle of column from straight line (eccentricity)
E Young's modulus
E_t tangent modulus
E_{eff} effective modulus
F total deflection of middle of elastic column
f non-dimensional form of F; $\rho f = F$
f^* frequency of periodic force
f_{st} value of f when inertia effects are disregarded (static)
f_1, f_2 non-dimensional displacement functions in perturbation
g, g^* non-dimensional additional deflection of column
H amplitude of periodic axial force
h parameter; $h = \Omega^{\frac{1}{2}}$
h^* depth of column section
I moment of inertia
I^* definite integral in Equation (33)
i index
j non-dimensional time; differs from ξ by an additive constant
K constant in stress-strain law
K^*, K^{**} parameters defined by equations (28) and (35)
k constant
L length of column
L' reduced length of column
L^* distance between end points of arch
ΔL shortening of distance between end points of column
ΔL_{comp} shortening of column because of compressive strain
ΔL_{curv} shortening of distance between end points of column because of curvature
ΔL_{spring} shortening of spring
ΔL_{tot} total shortening of system
M bending moment
m non-dimensional bending moment
$2m^*$ concentrated mass of model of column
n power in stress-strain law
P total axial force in column (positive sign indicates compression in Appendix II and tension in Appendix III)
p, p^* non-dimensional axial force (positive sign indicates tension)
Q total transverse load applied to arch
q non-dimensional transverse load applied to arch
R constant
R_a magnification factor
r radius of curvature of column
r^* radius of thin-walled circular cylindrical shell
ds element of length of column
T time elapsed between consecutive maxima of oscillations
t time
t^* wall thickness of circular cylindrical shell
u axial displacement
v non - dimensional transverse velocity; $v = da/dt$
w non-dimensional form of y; $\rho w = y$
w_{ad} non-dimensional form of y_{ad}; $\rho w_{ad} = y_{ad}$
w_0 non-dimensional form of y_0; $\rho w_0 = y_0$
X non-dimensional time elapsed between consecutive maxima of oscillations
x non - dimensional axial co-ordinate; $Lx = x^*$
x^* axial co-ordinate
y total lateral deflection of column
y_{ad} additional lateral deflection of initially curved column
y_0 initial deviation of axis of column from straight line
z variable
α perturbation of a
α^* coefficient in stress-strain law
$d\alpha^{**}$ angle subtended by sections of column ds apart
β perturbation of b
γ variable; $\gamma = \pi x^*/L$
ε strain (positive when tensile)
ε_c strain on concave side of column (compression side)
ε_E Euler strain; $\varepsilon_E = \pi^2(\rho/L)^2$
ε_{nom} nominal strain; corresponds to average stress
ε_0 reference strain
ε_t strain on convex side of column (tension side)
η perturbation of non-dimensional displacement
μ mass per unit volume
ξ non-dimensional displacement of testing machine; $\xi = ct/(L\varepsilon_E)$
Π force parameter
Π_{cr} value of force parameter at critical load
ρ radius of gyration of column section
σ stress (positive when tensile)
σ_{ave} average stress
σ_0 reference stress
τ duration of load
Φ dynamic similarity number defined in Equation (23a) and Equation (207)
φ perturbation of p
φ^* phase angle
χ function of ξ
χ^* integral of $\chi^{\frac{1}{2}}$ defined in Equation (97)
ψ amplitude of perturbation, function of time
Ω dynamic similarity number defined in Equation 9 and in Equation (76)
ω exponent in perturbation function, function of time
ω_n natural frequency of flexural vibration
∂ partial differential sign
$\mathscr{E}$ Considère's notation for effective modulus

1. The Development of the Theory of Columns

1.1. COLUMNS IN ANCIENT TIMES

The column is one of the oldest structural elements. At the beginning of recorded history, between 2500 and 2200 B.C., the Egyptians hewed columns out of rock in the tombs of Beni Hasan[1]. But in the minds of men in the Western World the word column is associated rather with the colonnades of ancient Greek and Roman public buildings. Perhaps the most famous of these, part of the Parthenon in Athens which was erected in the fifth century B.C., is shown in Fig. 1.

Apparently the columns of the classic period were designed entirely empirically, without the benefit of an analysis. To justify this assumption, a translation of three paragraphs of the book "De Architectura" by Vitruvius will be given. As far as is known Vitruvius Pollio was active as an architect during the reign of the Roman emperor Augustus, or possibly Titus. His book became famous and was reprinted, translated, and annotated many times during the Renaissance. The quotation is from Chapter I of Book 4 of the translation by W. Newton, Architect, published in London in 1791[2]. The original text is given in Appendix Ia[3]. Vitruvius speaks of the Greek colonists in Asia Minor:

> "There they began to erect fanes, and constitute temples to the immortal gods. First they erected the temple of Apollo Panionios, in the manner they had seen it in Achaia; which manner they called Doric, because they had seen it first used in the Dorian cities. In this temple they were desirous of using columns; but being ignorant of their symmetry, and of the proportions necessary to enable them to sustain the weight, and give them an handsome appearance, they measured the human foot of a man to be the sixth part of his height, they gave that proportion to their columns, making the thickness of the shaft at the base equal to the sixth part of the heighth, including the capital. Thus the Doric column, having the proportions, firmness, and beauty of the human body, first began to be used in buildings.
>
> "Afterwards, to construct the temple of Diana, they sought a new order from the same traces, copying the gracefulness of women, and making the thickness of the columns an eighth part of their height, in order to give them a taller appearance. Thus arose the invention of these two different orders; one of a masculine appearance, naked and unadorned; the other imitating the slenderness and fine proportion of women. But posterity, improving in ingenuity and judgment, and delighting in more graceful proportions, fixed the heighth of Doric columns at seven times their diameter; and of the Ionic, at eight and a half. This latter order was called Ionic, because it was first used by Ion.
>
> "The third which is called Corinthian, is in imitation of the delicacy of virgins; for in that tender age, the limbs are formed more slender and are more graceful in attire."

1.2. THEORY AND EXPERIMENT IN THE 18TH AND 19TH CENTURIES

This is the extent of the analysis of the stress and the stability of columns known to Vitruvius; and no progress could be recorded in this field until Leonhard Euler derived the formula which is still the cornerstone of our knowledge of the load-carrying capacity of columns. The history of this formula is related by Timoshenko in his "History of Strength of Materials"[4].

FIGURE 1. Colonnade of the Parthenon in Athens.

After Jacob and Daniel Bernoulli, the great mathematicians of Basle, Switzerland, derived the differential equations governing the deflections and the vibrations of beams, the latter wrote to his student Euler:

> "Since no one is so completely the master of the isoperimetric method* as you are, you will very easily solve the following problem in which it is required that $\int ds/r^2$ shall be a minimum."

The problem referred to is the determination of the elastic equilibrium of rods. Using the calculus of variations, Euler succeeded in deriving the least value of the axial compressive load under which the column is in equilibrium not only in its initial straight-line configuration, but also in a neighbouring, slightly bent configuration. The theory was published in the appendix "De curvis elasticis" to his book "Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes" in 1744[5]. Part of the text of the English translation by Oldfather, Ellis, and Brown[6] reads:

> "Therefore, unless the load to be borne be greater than $\pi^2 EI/4L^2$† there will be absolutely no fear of bending; on the other hand if the weight P be greater, the

* The calculus of variations.

† The symbols in this formula as well as in all the others to follow are those used in the present paper, not necessarily the same as the ones used by the original authors.

column will be unable to resist bending. Now when the elasticity of the column and likewise its thickness remain the same, the weight P which it can carry without danger will be inversely proportional to the square of the height of the column and a column twice as high will be able to bear only one-fourth of the load." (The original text is given in Appendix I*b*.)

It should be added that L is the length of a column one end of which is rigidly fixed while the other is free.

Today it is more usual to write the Euler formula for a column which has both of its end points simply supported. In this case the formula is

$$P_E = \pi^2 EI/L^2 \quad . \quad . \quad . \quad (1)$$

where P_E is the critical load according to Euler, E is Young's modulus, I the moment of inertia of the cross section, and L the length of the column.

In the nineteenth century Euler's formula was not often applied by engineers because it was found to lead to unsafe design. In 1889 Considère pointed out that Euler's theory was based on the assumption of perfect elasticity and could not possibly apply when the stress in any portion of the column exceeded the limit of elasticity. He made tests with 32 columns provided with adjustable knife-edge fittings and centred the columns after measuring their deflections carefully at half the buckling load. Thus Considère established the foundations of the modern techniques in column testing. He also clearly understood the concept of the effective modulus E_{eff}, although he was unable to derive an analytical expression for it. This can be seen from the following quotation from his lecture "Résistance des pièces comprimées,"[7] presented at the Congrès International des Procédés de Construction held in Paris, France, in 1889 (for the original text *see* Appendix I*c*):

> "In determining the value of E_{eff}, one has to distinguish between the outer and inner fibres relative to the curvature of the compressed column.
>
> "In the former, the curvature causes an elongation, and consequently a decrease in the compression; it is known that in such a case, namely when the material returns to the unstrained state, the metal behaves as if it were perfectly elastic. Thus for the outer fibres E_{eff} is equal to E, the modulus of elasticity."

For the inner fibres,

> "E_{eff} is equal to the slope of the tangent at point P.
>
> "As the total bending moment in the column results from the simultaneous action of the convex and concave fibres the value of E_{eff} . . . must be somewhere between those of the constant E and the slope of the tangent to the stress-strain curve. . . ."

The formula corresponding to the ideas thus expressed would be

$$P_{eff} = \pi^2 E_{eff} I/L^2 \quad . \quad . \quad . \quad (2)$$

where P_{eff} is the critical load of the column according to the effective modulus concept.

Obviously, Professor Friedrich Engesser of the Polytechnic Institute of Karlsruhe, Germany, did not know of Considère's work when in 1889 he suggested[8] that the buckling load of a column should be calculated from a modified Euler formula, with Young's modulus replaced by the tangent modulus E_t, which is the slope $(d\sigma/d\varepsilon)$ of the stress-strain curve corresponding to the average stress at buckling:

$$P_t = \pi^2 E_t I/L^2. \quad . \quad . \quad . \quad (3)$$

Here P_t is the critical load of the column according to the tangent modulus concept. His mistake in not taking into account the difference in the behaviour of the material in loading and in unloading was pointed out by Professor F. Jasinski of St. Petersburg, Russia[9], who made reference to Considère's work and wrote in the June 1895 issue of the *Schweizerische Bauzeitung* (for the original text *see* Appendix I*d*):

> "Of course it is always possible to express the magnitude of the buckling load of a column in which the elastic limit is exceeded by the formula
>
> $$P = 10 E_{eff} I/L^2 \quad . \quad . \quad . \quad (2a)$$
>
> where E_{eff} is some unknown function of the dimensions of the column, but this function is not equal to, but greater than, $d\sigma/d\varepsilon$. For the time being it is impossible to derive theoretically the form of this function, and thus only the empirical approach is available. Naturally it would be easy to determine the value of E_{eff} from the experiments of Tetmajer, but through such a procedure equation (2*a*) would become an unnecessarily complicated version of the Tetmajer formula."

Engesser soon accomplished the impossible. In his answer to Jasinski printed in the July 1895 issue of the *Schweizerische Bauzeitung*[10] he gave the correct general formula for the reduced modulus of an arbitrary cross section based on the same assumptions as those expressed by Considère in the foregoing quotation. He also remarked (for the original text *see* Appendix I*e*):

> "According to the preceding equation P_{eff} appears to depend not only on I, but also on a more complex function of the cross section; cross sections of the same moment of inertia but of different shape do not yield, therefore, the same buckling strength. A similar situation exists in regard to the ultimate strength in flexure."

1.3. BUCKLING THEORY AND PRACTICE AT THE BEGINNING OF THE 20TH CENTURY

Apparently the work of Considère and the Engesser-Jasinski controversy did not attract the attention of the engineering world. One must draw this conclusion from the lack of any mention of the names or the work of these scientists in the comprehensive book on buckling published by L. von Tetmajer in 1903 under the title "Die Gesetze der Knickungs-und der zusammengesetzten Druckfestigkeit der technisch wichtigsten Baustoffe"[11]. As Tetmajer's experiments were done in Zurich, Switzerland, while he was professor at the Federal Polytechnic Institute, he was in the best position to read and appreciate the articles appearing on buckling in the Swiss journal *Schweizerische Bauzeitung*. Both Tetmajer's book "Die angewandte Elastizitäts-und Festigkeitslehre"[12] which appeared in 1904 and the widely used volume "Elastizität und Festigkeit" by C. Bach[13] published in 1902, discuss only the Eulerian theory of perfectly elastic columns and contain empirical formulae for the load-carrying capacity of short columns.

In England the treatise "The Design of Columns and Struts"[14] by W. E. Lilly (1908) presents only the Euler theory, the Rankine-Gordon formula, and a great deal of experimental information; and Arthur Morley[15] in his "Strength of Materials" (1908) makes no mention either of Considère, Engesser, or von Kármán. In the

United States even the sixth edition of " The Theory and Practice of Modern Framed Structures "[16] published by J. B. Johnson, C. W. Bryan, and F. E. Turneaure in 1916 has no theory for the buckling of short columns.

1.4. COLUMN STRENGTH AND THE WRIGHT BROTHERS

It can be concluded, therefore, that the theory of perfectly elastic (long) columns was widely known at the time when Wilbur and Orville Wright designed and constructed their first gliders and powered aeroplanes. Although such a rational theory of the inelastic buckling of short columns was not generally available, practical engineers had enough empirical information to permit them to design safe columns without undue weight. This was of great importance to the Wright brothers because the wings of their biplanes were connected by many struts. The struts were attached to the wing spars by means of simple universal joints to permit the warping of the wings (*see* Fig. 2); as is well known this wing warping was the Wrights' device to control the rotation of the aeroplane about the longitudinal axis.

The device was explained by Orville Wright in the following manner in a deposition made in the Court of Claims of the United States[17] on 2nd February 1921:

> " 28 *Cross-question.* How was it that this relative raising and lowering could take place when the diagonal trussing wires were present?
>
> " *Answer.* The upper and lower surfaces were connected to one-another by means of uprights having universal joints at their connections with the surfaces.
>
> " These universal joints permitted the upper and the lower planes to be brought into diagonal positions with reference to each other without distorting the lateral trussing of the forward and rearward systems of uprights and spars."

At the beginning of this century structural design was a well developed art while aerodynamics was in its infancy and naturally next to nothing was known about the stability and the control of aeroplanes. For this reason Wilbur and Orville Wright devoted most of their energies to the solution of the aerodynamic problems and did little to improve the theory of structures. This does not mean, however, that they took the problems of structural safety lightly. A few quotations from the yet unpublished Wright Papers (published December 1953 —*Ed.*) in the Library of Congress in Washington, D.C., will prove this statement.

On 15th December 1934 Orville Wright wrote to Willard A. Driggers:

> " In answer to your inquiry as to whether there was any stress analysis of the original Wright aeroplane, and whether early aircraft constructors attempted strength calculations, will say that there was a stress analysis made of our first power machine as well as of the gliders which preceded it. These analyses, however, were not so complete as those made today.
>
> " I suspect that stress analyses were made of Chanute's gliders as Mr. Chanute was an engineer and bridge builder of high standing. As to the other early constructors I have no information."

The Wrights were most careful in selecting and testing the materials of construction and considered dynamic tests as important as static tests. On 6th September 1917 Orville Wright wrote to Thomas R. C. Wilson:

> " In 1901, 1902, and 1903 we made a number of tests upon Eastern spruce and white pine. The spruces when kiln dried by the processes of that day were very inferior to the air-dried. The white pine did not seem to suffer so much in the dry kiln. The shock tests indicated that the spruce was superior to clear white pine. But the superiority of the spruce was more apparent in practical use in the machines."

The shock test is described in a letter Orville Wright wrote to F. O. Clements on 17th July 1917:

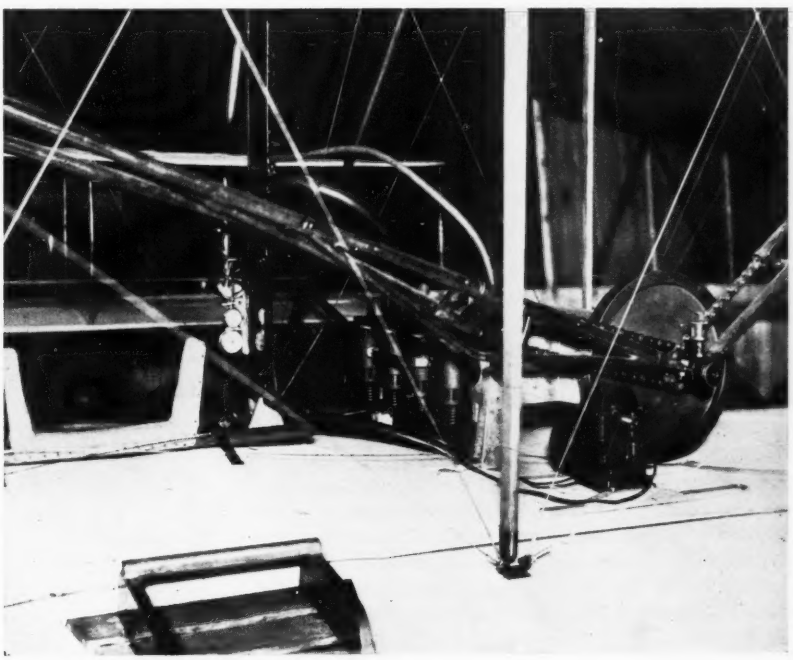


FIGURE 2. Interplane strut of biplane flown by Wright Brothers in Kitty Hawk on 17th December 1903.

(Courtesy of Smithsonian Institution)

"I am giving below the results of the tests made on the several pieces of lumber which you sent me.

"The pieces tested were 1¼ in. square. A blow with a 21½ lb. wooden hammer was applied midway between the supports which were 20 in. apart."

On 23rd September 1900 Wilbur Wright tried to convince his father, the Reverend Milton Wright, that he was taking the necessary precautions to ensure that their glider was safe structurally:

"I am constructing my machine to sustain about five times my weight (700 lb. plus?) and am testing every piece."

Regarding the strength tests of the complete aeroplane one may refer to an entry in Orville Wright's diary made on 16th November 1903:

"We hung the machine on tips to test strength. The cloth wrinkled badly, necessitating a change in the trussing of the tips so that the strain be more evenly carried by front and rear spars and uprights. This requires a change in the controlling wires which will have to be operated at the rear. The uprights stood test of twice what will be required of them even though the wire connecting their centers had not been put in yet."

An entry on 17th November 1903, reads:

"Worked today on changing wires and in putting on wire to connect centers of front uprights. Supported at tips machine stood test of 440 lbs. at center without any signs of being overburdened."

Finally one might quote from a letter Wilbur Wright wrote just two weeks before the first successful human flight in a powered heavier-than-air machine. The letter is addressed to George A. Spratt and is dated 2nd December 1903 from Kill Devil Hills:

"We have not tried to glide the big machine yet, and probably shall not this year, as favorable days are very scarce now. We hung it on its wing tips some days ago and loaded the front set of trussing to more than six times its regular strain in the air. We also hung it by the tips and ran the engine screws with the man also on board. The strength of the machine seems 'O.K.'"

1.5. THE ESTABLISHMENT OF A RATIONAL THEORY OF SHORT COLUMNS

The feasibility of a rational formula in the calculation of the buckling load of columns which fail at stresses exceeding the elastic limit of the material was generally recognised only after the results of Th. von Kármán's investigations were published in 1910. The paper was a doctoral dissertation at the University of Göttingen with L. Prandtl as thesis adviser. The title is "Untersuchungen über Knickfestigkeit"[18]. At the time when he worked on his dissertation von Kármán knew of Engesser's first paper and of Considère's work, but had not seen Engesser's improved theory.

Von Kármán calculated the deflections of, and the stresses in, columns loaded in axial compression for various amounts of the eccentricity of the loads. In establishing static equilibrium, he took into account the experimentally established stress-strain relationship of the material. He arrived at families of curves showing the connection between load and maximum deflection with the eccentricity as the parameter. On p. 18 of his publication he discusses the curves obtained for inelastic buckling in the following manner (for the original text *see* Appendix I*f*):

"These figures show clearly the effect of the eccentricity upon the maximum load and the history of the process after buckling. In particular it can be seen that even a very minor eccentricity decreases materially the maximum load while in the perfectly elastic case the curves corresponding to the various values of the eccentricity all approach asymptotically the same value.

"This phenomenon, which is of great importance in practical applications, is a consequence of the fact that the deflections cause a rapid drop in the load when buckling takes place above the elastic limit while in the perfectly elastic buckling process the deflections increase under an almost constant load. . . . It will be shown later that these theoretical conclusions are corroborated by the experiments."

On the assumption of small deformations von Kármán derived a simple formula for the buckling load of a short column (equation 2); he was the first to give simple closed-form expressions for the reduced modulus of the solid rectangular and the idealised I-sections. He then proceeded to the experimental verification of his theory. The major features of the experimental work were the careful determination of the compressive stress-strain diagram of the material of the column and the accurate centring of the specimens under load by means of adjustable knife-edge end fittings. He also made rational corrections in the evaluation of the test results to allow for the increased rigidity of the column in the region of the end fittings.

The maximum load recorded in the tests was almost always between the values calculated from the theory for no eccentricity and for an eccentricity of 0·005 times the depth of the cross section. This was true for both the long and the short columns within a range of the slenderness ratio L/ρ from 22 to 176. The results established the correctness of the Euler-von Kármán theory of the buckling of columns.

The Engesser-von Kármán formula was rederived independently by Sir Richard Southwell in 1912[19]. He compared his theoretical formula with the experiments of Dr. Lilly[14]. Instead of using an effective modulus E_{eff}, Southwell proposed a reduced length L'. To quote from the paper:

"We shall, therefore, assume in our theory that when the material has been uniformly stressed beyond the elastic limit, it has two moduli of elasticity, the first, corresponding to a decrease of stress, equal to E, the original modulus of the material; and the second, corresponding to an increase of the stress, having a much lower value E_t, which may be obtained from the stress-strain curve.

"The extent to which the value of L'/L depends upon the shape of the cross-section is illustrated by Fig. 5*, in which the relation between L'/L and E_t/E is shown graphically for solid circular, thin tubular, and square sections.

"It may be seen from this diagram that the employment of equation (6)* for any of these types of section will not introduce any great inaccuracy; but there is a sensible difference in the case of short struts, and it is clear that any new type of section ought to receive separate treatment."

The differences to which Sir Richard refers are those between the curves representing the solid circular, thin tubular, and solid square sections.

* This figure and this equation are not reproduced in the present paper.

1.6. RETURN TO THE TANGENT MODULUS LOAD

In 1946 Francis R. Shanley[20], then Division Engineer in Charge of Engineering Research at the Lockheed Aircraft Corporation, suggested that the tangent-modulus load given in equation (3) be accepted as the correct buckling load instead of the effective modulus load of equation (2). He argued that a column would buckle under the load P_t if the compressive strain increased, or remained constant, in every point of the cross section during the buckling process; under these conditions naturally the stress-strain relationship is governed by E_t as long as all the deflections are small. But the condition stated can be fulfilled in a buckling test if the flexure of the column takes place simultaneously with the necessary increase in the average compression.

In a discussion of Shanley's suggestion it is important to realise that the system upon which his reasoning is based differs considerably from that of the classical analysis. In the Euler-von Kármán system the end points of the column are subjected to a fixed axial compressive load, while in the Shanley system the end load is not prescribed, but keeps changing during the buckling process.

If the buckling load is defined in the Eulerian sense, it is the load under which a system is in equilibrium, not only in a single configuration of the deformations but also in neighbouring configurations. The system is composed of a solid as well as of its geometric constraints and the forces prescribed over part of its boundary. With this definition, the buckling load of a short column is given uniquely by the von Kármán formula of equation (2).

It is more difficult to find out what the buckling load is in a column test. Yet this information is of greater immediate interest to the practical engineer than the derivation of an expression for the critical load corresponding to the classical theory. At the same time one should determine whether the conditions prevailing in the column test are truly representative of the loading of a building in a wind storm or during an earthquake, and of an aeroplane in a gust or at landing. It is conceivable that the compressed elements of the structure buckle under different loads under these various conditions. To investigate all these ramifications of the buckling problem one has to analyse the dynamics of the motion of the column during the buckling process. In the next Section this will be done for the column in the testing machine.

It can be stated, however, even before such an analysis is undertaken, that the acceptance of the tangent modulus load as the correct buckling load is equivalent to the rejection of the classical concept of stability. It is Shanley's merit to have pointed out that a short column can buckle in an ordinary testing machine at a load which is noticeably smaller than the critical load of the classical theory.

2. Dynamic Investigation of the Ordinary Column Test

2.1. IMPERFECT COLUMNS

In an investigation of the process of buckling in the testing machine, the imperfections of the column must be taken into account. This is necessary because perfect columns do not exist; the machinist is only capable of manufacturing a column within given tolerances, and the test engineer can only centre it imperfectly. But the assumption of deviations from symmetry is indispensable for one more reason: Without a disturbance of the perfect symmetry, the column must remain perfectly symmetric during the entire loading process. It only shortens elastically as the load is increased if its material is perfectly elastic. A column made of an elasto-plastic material can be compressed to a pancake and it will still remain symmetric.

In the classical Eulerian approach perfect columns are treated. However, the stability limit is established by investigating the effect of a slight displacement from the initial configuration of equilibrium. In a similar manner, the dynamic analysis can be carried out for a perfect column through calculation of the motion following a disturbance in the equilibrium configuration. Indeed this was done by V. L. Salerno, Frances Bauer, and James Sheng at the Polytechnic Institute of Brooklyn[21].

In the present paper, a different approach is followed. It is assumed that the centre line of the column in its initial, natural, unstressed state deviates slightly from the straight line. Even if the deviations are so small that they cannot be measured directly in the laboratory, their effect is marked both in the analysis and in an experiment.

2.2. EQUATIONS OF MOTION OF THE PERFECTLY ELASTIC COLUMN

No controversy has yet arisen on the behaviour of a perfectly elastic column. Nevertheless, its buckling process is analysed here in order to establish theoretically the dependence of its deflections and of the compressive force acting along its axis on the displacement of the loading head of the testing machine. The theoretical results will be compared in a later section with measurements made in experiment. The analysis also serves as a basis for the investigation of the buckling process of columns failing at stresses greater than the elastic limit of the material.

Figure 3 shows a perfectly elastic column in the testing machine. The column is compressed by the downward displacement of the loading head. It is reasonable to represent the actual conditions prevailing in the column test by assuming that the downward velocity c of the loading head is prescribed. This assumption is certainly much closer to reality than that in which the variation of the load is stipulated.

The motion of the loading head initiates pressure waves which travel in the direction of the axis of the column and are reflected at both ends. As the velocity of these waves is about 200,000 in./sec., the variations they cause in the axial distribution of the pressure may be disregarded in the present analysis, as is indicated in Appendix II and in earlier publications[22]. The compressive force P in the column is therefore assumed to be independent of the axial co-ordinate x^* and only a function of time t.

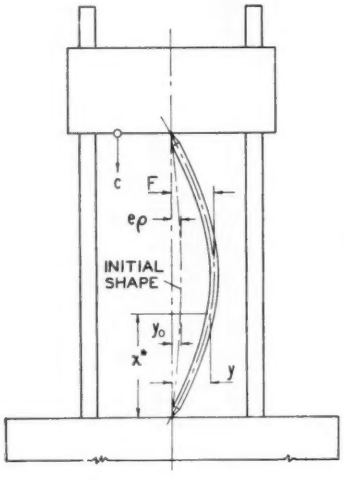


FIGURE 3. Column in testing machine.

Under these conditions Newton's equation of motion can be written for the transverse motion of an element of the column:

$$(\partial^2/\partial x^{*2})\{EI\,[\partial^2 y/\partial x^{*2})-(\partial^2 y_0/\partial x^{*2})]\}+P\,(\partial^2 y/\partial x^{*2})+ +\mu A\,(\partial^2 y/\partial t^2)=0 \quad . \quad . \quad . \quad (4)$$

where EI is the bending rigidity of the column, A its cross-sectional area, μ the mass per unit volume of its material, y the lateral displacement from the straight line connecting the two knife edges, and y_0 the value of y at the beginning of the loading process when the column is unstrained.

The compressive axial force in the column would be $(EA/L)\,ct$, that is the displacement ct at time t divided by the length L of the column and multiplied by Young's modulus E and the cross-sectional area A if the column were not relieved as it deflected laterally. The relief in strain is the difference between the lengths of the column in the deflected position characterised by y and in the initial position defined by y_0, divided by the initial length L. The relief can be easily calculated from the geometry of the deformations. One obtains for the compressive force

$$P=(EA/L)\left\{ct-(1/2)\int_0^L[(\partial y/\partial x^*)^2-(\partial y_0/\partial x^*)^2]\,dx^*\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If it is now assumed that the cross section is constant and the initial shape is a simple sine wave, it is readily seen that the shape of the deflections always remains the same and only the amplitude changes with time. The statement can be proved by assuming a product solution with one factor the sine function in x^* and the other an unknown function of time. When this is substituted in equations (4) and (5), the sine function appears as a common multiplier of terms which are functions of time alone. The solution holds if the time function satisfies an ordinary non-linear differential equation. This equation is derived in Appendix II (equation (75)).

2.3. THE ELASTIC BUCKLING PROCESS

The motion of the column during the buckling process is known if the amplitude F of the sinusoidal deflections shown in Fig. 3 is obtained as a function of time. In the analysis of Appendix II a non-dimensional amplitude f is introduced by dividing F by the radius of gyration ρ of the cross section. Similarly, time is non-dimensionalised as f is treated as a function of ξ which is the loading head displacement ct divided by the column length L and the Euler strain $\varepsilon_E=\pi^2(\rho/L)^2$:

$$\xi=ct/L\varepsilon_E \quad . \quad . \quad . \quad (6)$$

Solution of equation (75) presents serious difficulties. For the purpose of the investigations described in this Section, the following approach appears advantageous. One can first solve the equation neglecting the inertia term and then can try to evaluate the modifications of the motion caused by that term. The procedure is promising because the loading head descends so slowly in routine tests that the inertia effect should be a secondary one.

The static solution f_{st} is easily calculated (*see* equations (79) and (80)). The values obtained for f_{st} and for P are plotted in Fig. 4 against ξ for two values of the initial deviation amplitude $e\rho$. The graph reveals that a column with very small initial deviations from straightness deflects little at the beginning of the loading process. When the displacement of the loading head approaches the value characterised by $\xi=1$, the deflections increase rather suddenly. The value $\xi=1$ has a physical significance; it corresponds to a loading head displacement ct equal to the shortening $\varepsilon_E L$ of a perfectly straight column under the action of the Euler load. For this reason the displacement at $\xi=1$ will be referred to as the Euler displacement.

The initially less straight column shows gradually increasing deflections which are considerably in excess of those of the straighter column, but the two deflection curves approach each other after the Euler displacement is surpassed. The load increases practically proportionately to the loading head displacement when $e=0{\cdot}001$ until the Euler load is reached and remains constant from there on. With the less straight column,

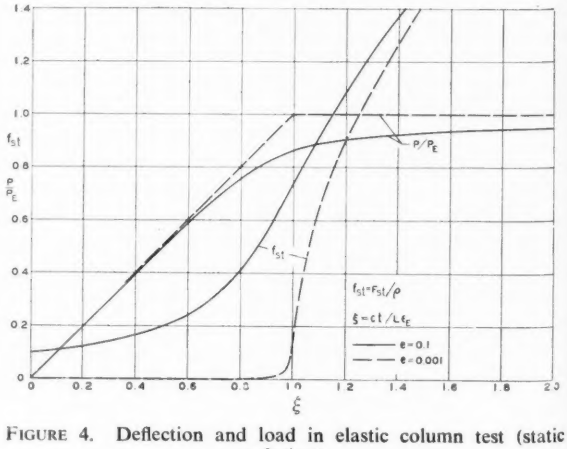


FIGURE 4. Deflection and load in elastic column test (static solution).

the load lags behind noticeably after half the Euler load is reached; the value of P/P_E approaches unity asymptotically as the loading head displacement increases.

2.4. VIBRATIONS SUPERIMPOSED UPON THE STATIC SOLUTION

It is of considerable importance to determine how closely the static solution illustrated in Fig. 4 resembles the actual behaviour of the column under the dynamic conditions of the column test. This was done in Appendix II by developing a perturbation solution of equation (75). The analysis presented there follows closely an unpublished solution of the problem communicated to the author by Professor George F. Carrier in a letter in which he discussed the second of References 22 in 1950.

In the perturbation approach it is assumed that the dynamic solution f differs little from the static solution f_{st} and

$$f=f_{st}+\eta \quad . \quad . \quad . \quad (7)$$

In this equation η is an unknown function of time. Substitution of the right-hand member of this equation in equation (75) (which is the non-dimensional form of equation (4)) yields

$$\eta''+h^2\chi\eta=-f_{st}'' \quad . \quad . \quad (8)$$

if terms containing the second and third powers of η are rejected as small of a higher order. In equation (8) the constant

$$h^2=\Omega=\pi^2\varepsilon_E{}^3\,(E/\mu c^2)=\pi^8\,(a_s/c)^2\,(\rho/L)^6$$
$$=\pi^2\varepsilon_E\,[\omega_n/(c/\rho)]^2 \quad . \quad (9)$$

is the dynamic similarity number which governs the dynamics of the buckling process. It depends only on the geometry and the mechanical properties of the material of the column. In equation (9), $a_s=(E/\mu)^{\frac{1}{2}}$ is the velocity of sound in the material of the column and ω_n is the natural frequency of its small flexural vibrations. The value of h^2 in a routine test of a perfectly elastic column is usually between 10^7 and 10^9.

The symbol χ designates a known function of ξ, the independent variable (*see* equation (84)), and the primes indicate differentiation with respect to ξ.

The first simplification in the solution consists of disregarding f_{st}'' and thus reducing equation (8) to a homogeneous equation. Next, following the B.W.K. method[23], the solution is assumed in an exponential form with an exponent which is a function of time. This solution satisfies equation (8) (with $f_{st}''=0$) approximately, and the approximation becomes better as the value of $h^2=\Omega$ increases. A rule derived for the validity of the approximate solution is

$$6{\cdot}5\,e^2h^2 \gg 1 \quad . \quad . \quad . \quad (10)$$

This inequality, given in Appendix II as equation (144), is always satisfied in ordinary column tests. The deviation of the axis of the column from straightness is not likely to be less than a thousandth of the radius of gyration. Hence $e \geqslant 10^{-3}$ and thus the inequality requires that $h^2 \geqslant 10^7$. But the dynamic similarity number defined in equation (9) usually exceeds 10^7.

The expression obtained for η is given as equation (96). It represents a vibration whose amplitude varies from C^* at $\xi=0$ to not more than $3C^*$ near $\xi=1$ (*see* equation (109)). The frequency of the vibrations about the static deflections changes during the process of buckling; the order of magnitude of the period is a hundredth of a second (*see* equation (122)).

It can be concluded, therefore, that the static solution is a satisfactory representation of the consecutive positions assumed by an elastic column as the loading head of the machine descends in a column test. If some disturbance occurs in the static equilibrium during the test, because of vibrations of the foundation or the jar with which the machine is started, the amplitude of this disturbance will not be magnified more than by a factor of 3.

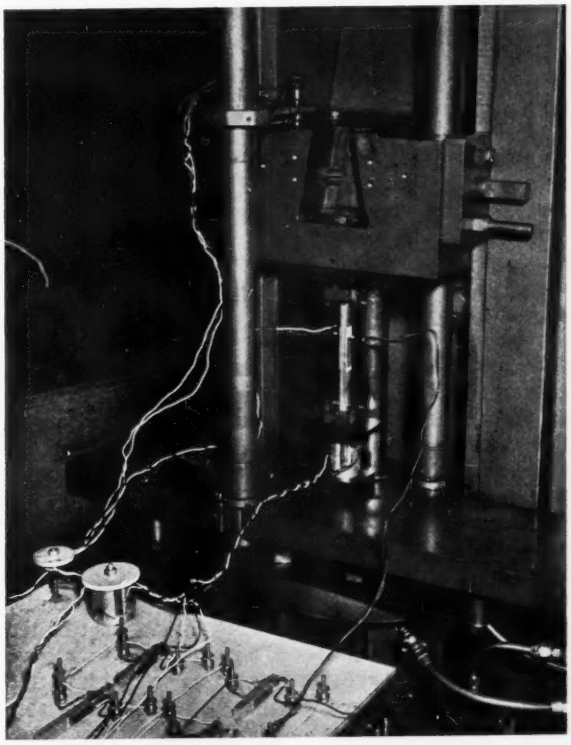
Of course, the static solution does not represent correctly the initial conditions. It should normally be assumed that the non-dimensional transverse velocity $df_{st}/d\xi=f_{st}'$ is zero when $\xi=0$. But equation (101) yields upon substitution of $f_{st}=e$

$$(f_{st}')_0=2e^2/(2e+e^3) \simeq e \quad . \quad (11)$$

This is a small non-dimensional velocity, but it is not zero. If it were considered an initial disturbance and introduced in the perturbation solution, the factor C^* in equation (94) would become

$$C^*=e/h \quad . \quad . \quad . \quad (12)$$

Oscillations of this magnitude can hardly be observed;



(*Courtesy of Society for Experimental Stress Analysis*)

FIGURE 5. Test arrangement in hydraulic machine.

they are also rapidly damped out by the friction at the knife edges and by internal damping.

Finally the omission of the term f_{st}'' from equation (8) must be justified. It is shown in Appendix II that this term is negligibly small compared to η'' if the order of magnitude of the oscillations following a disturbance is e, provided that h^2 exceeds a value of about 10^7 (*see* equation (133)). This means that f_{st}'' can give rise to vibrations of an amplitude considerably smaller than e. Such vibrations are of no practical interest.

2.5. EXPERIMENTS WITH COLUMNS

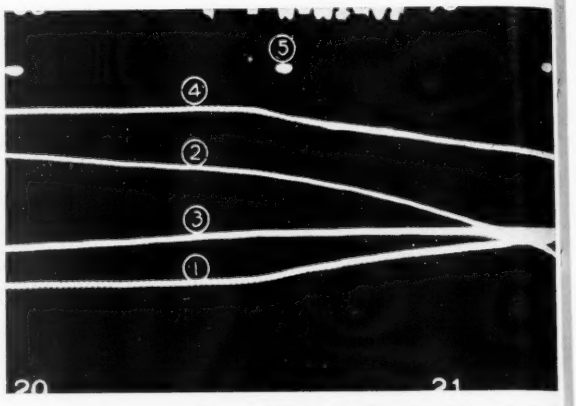
Experiments with columns were made in the Airplane Structures Laboratory of the Polytechnic Institute of Brooklyn to check the predictions of the theory just discussed and to compare the behaviour of long columns in the testing machine with that of short columns. The results were published by the author jointly with S. V. Nardo and Burton Erickson[24].

The test arrangement is shown in Fig. 5. The column was provided with adjustable knife edges and was placed in a Sonntag-type Baldwin Southwark Universal Testing Machine of 12,000 lb. load capacity. The column was aligned vertically with the aid of a surveyor's transit and centred under load in the customary fashion with the aid of Baldwin-Southwark SR-4 strain gauges cemented to opposite sides of the centre of the column. The cross section of the column was a solid rectangle of $\frac{1}{2}$ in. by $\frac{3}{8}$ in.; its material was 24S-T aluminium alloy. An effort was made in each test to run the hydraulic testing machine at a constant velocity of the loading head by adjusting the valves manually, and the velocity was varied from test to test between 0·002 and 0·05 in./sec. Within this range no effect of the velocity upon the behaviour of the column could be observed.

During the test the time history of the downward displacement of the loading head, of the axial compression in the column, and of the strain on the concave and the convex sides of the middle of the column were recorded. The strains were measured by the two bonded wire strain gauges already mentioned. For the measurement of the head displacement a leaf spring, and for that of the compressive force a thin-walled cylinder, were used and both were equipped with SR-4 strain gauges. The signals received from these gauges were amplified in a four-gun oscilloscope constructed at the Polytechnic Institute of Brooklyn and were photographed with a Fairchild continuous-strip motion picture camera.

The test results obtained with a slender column are shown in Fig. 6. The traces are, from top to bottom, in the order they appear at the left edge of the figure: The compressive strain on the convex side (this quantity increases upward); the downward displacement of the loading head (this increases downward); the compressive force in the column (this increases upward); and the compressive strain on the concave side (this increases upward). The dots along the upper edge of the record are time marks spaced 0·75 sec. apart.

Initially the two strains are equal; they are set apart in the picture for a better utilisation of the width of the



(Courtesy of Society for Experimental Stress Analysis)

FIGURE 6. Oscillogram of test in hydraulic machine with slender column. The traces are: (1) strain on concave side; (2) head displacement; (3) axial force in column; and (4) strain on convex side; (5) is a time mark. The initial values of the strains are equal. They are projected apart for the sake of clarity.

oscilloscope. The trace of the strain on the concave side of the column begins at the lower edge of the oscillogram because it rises rapidly when the column buckles, while the strain on the convex side is projected near the upper edge to have the full width of the oscilloscope available when it starts to move downward.

The photograph reveals that all the four traces are straight lines until the buckling load is reached. At that moment a break can be observed in the lines, and from there on the strain on the concave side of the column increases at a higher rate and that on the convex side at a lower rate than before. The compressive load remains constant, and the loading head velocity increases. Except for the last item, all the variations recorded are in good agreement with expectation based on the Euler buckling theory. The increase in the loading head velocity is mainly a consequence of the elasticity of the testing machine. Under a constant load the pumps must furnish only enough fluid to take care of the increased volume corresponding to the displacement of the loading head, while before the Euler load is reached they also have to fill the additional space created by the elastic expansion of the pipes and containers under the increasing oil pressure.

The results of a test with another slender column were evaluated from an enlargement of the oscillogram and were plotted in Fig. 7. The curves are similar to those shown in Fig. 6. The slight decrease in the compressive load after the maximum is reached can be attributed to the compressive stress exceeding the elastic limit of the material.

In Fig. 8, the history of the load and strain of a slender column is shown as calculated from the theoretical equations (79) and (80). The agreement between Figs. 7 and 8 is satisfactory. The slight decrease in the load in Fig. 7 is naturally not reproduced in Fig. 8 because in the theory the material is assumed to be perfectly elastic. Another difference between the two plots is the more rapid increase in strain at the end of the loading process in the experimental than in the

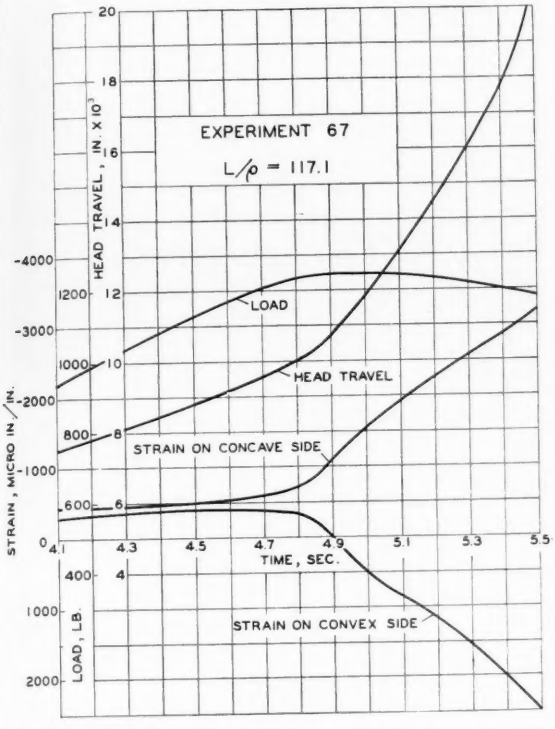


FIGURE 7. Results of buckling test with slender column in hydraulic testing machine.

theoretical curves. This is a consequence of the increase in speed of the loading head of the testing machine which was discussed earlier; in the theory the velocity c is assumed to be constant. Finally, it should be remembered that the slenderness ratio of the columns represented in the two figures is not the same and that the eccentricity e of the test column was not measured.

Figure 9 is a reproduction of the oscillogram obtained with a short column. The arrangement of the traces is the same as in Fig. 6 except for an interchange of the loading head displacement and the load records. However, the behaviour of the column is radically different. When the buckling load is reached, violent changes take place in all the quantities measured. The column snaps from its slightly deflected position into a highly deflected one with an audible thud. Records taken at higher camera speed have shown that, on the average, the snap takes about one hundredth of a second.

2.6. DEVELOPMENT OF A THEORY FOR SHORT COLUMNS

One of the objectives in developing a dynamic theory of the buckling process of short columns was the explanation of the snap action. In 34 tests performed in the first series of experiments described in Reference 24 every short column snapped through and every long column failed gradually. It was not unreasonable to assume, therefore, that the violent changes taking place at buckling were a consequence of the non-linearity of the stress-strain relationship beyond the elastic limit. Nevertheless, several attempts to obtain the jump theoretically through introduction of a non-linear stress-strain law failed completely. It appeared from the calculations that the relief in compression accompanying the transverse deflections was so great that it effectively stopped any incipient sudden lateral motion.

This was found to be true when the displacement of the loading head was prescribed. The remedy for the situation was the assumption of some elasticity in the testing machine itself; this elasticity permitted the loading head to follow the downward movement of the upper end of the column without a substantial loss in the compressive force when the column deflected laterally.

When this idea was introduced in the theory, the jump phenomenon was obtained. The pre-requisites for the snap-through (or jump) process are, therefore, a non-linear stress-strain relationship and an elastic testing machine. If either feature is missing, buckling occurs gradually.

It is of some interest to note that true inelastic behaviour is not required for the jump phenomenon, and a non-linear but perfectly elastic stress-strain law suffices for its explanation. This is fortunate, because the permanent deformations developing above the elastic limit of actual metals give rise to different laws for loading and unloading, which are troublesome in an analysis of the buckling process.

In the calculations to follow, the material is therefore assumed to obey the same non-linear stress-strain law in both loading and unloading. A real column will behave as predicted by this theory as long as the compressive strain keeps increasing everywhere in the material. The theory is also capable of predicting correctly the occurrence of a strain reversal, but beyond that it represents only an approximate solution of the actual problem.

2.7. STATIC EQUILIBRIUM OF NON-LINEAR COLUMN

The stress-strain law selected for the analysis of the non-linear column is

$$(\varepsilon/\varepsilon_0)=(\sigma/\sigma_0)^n \quad . \quad . \quad . \quad (13)$$

It is shown diagrammatically in Fig. 10 for three different odd integral values of n. Such curves differ significantly from experimental diagrams obtained with structural materials as all of them, except the one for

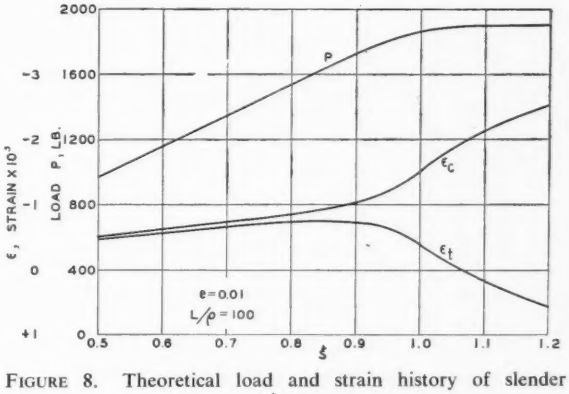
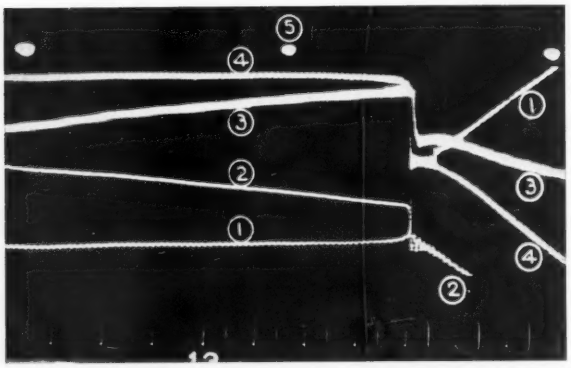


FIGURE 8. Theoretical load and strain history of slender column.



(*Courtesy of Society for Experimental Stress Analysis*)

FIGURE 9. Oscillogram of test in hydraulic machine with short column. The initial values of the strains are equal. They are projected apart for the sake of clarity. The traces are numbered as in Fig. 6.

which $n=1$, have a vertical tangent at $\sigma=0$. However, in a restricted range of the strain, the stress-strain law can represent the actual conditions with a satisfactory degree of accuracy. This is particularly true when n is chosen as 9 or 11, but the mathematical difficulties accompanying such a choice are too great. It was decided therefore to forego the greater accuracy in favour of the ease of calculation and to base the analysis on a stress-strain law in which the exponent is 3. With the numerical constant selected, the final form of the stress-strain relationship is

$$\varepsilon=(\sigma/K)^n=(\sigma/300{,}000)^3 \quad . \quad . \quad (14)$$

This relationship is shown in Fig. 11. In Fig. 12 the tangent modulus

$$E_t=d\sigma/d\varepsilon \quad . \quad . \quad . \quad (15)$$

is plotted against the stress and Fig. 13 is the column curve which shows the buckling stress as a function of

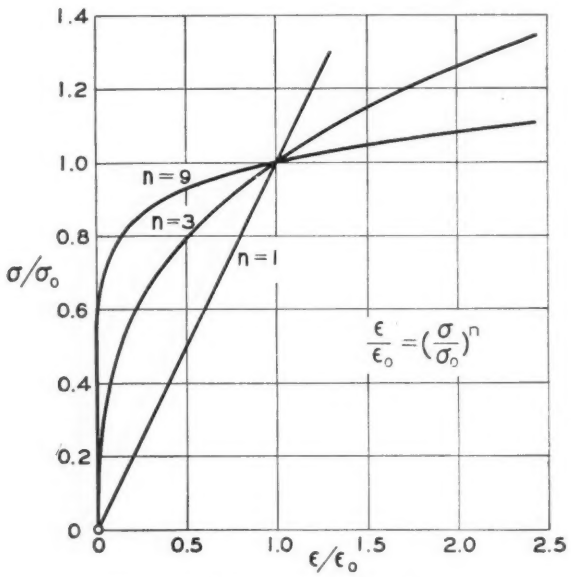


FIGURE 10. Non-linear stress-strain law.

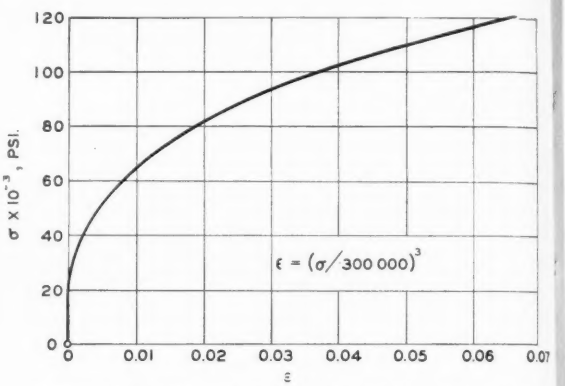


FIGURE 11. Cubic stress-strain law underlying the analysis.

the slenderness ratio L/ρ of the column. Details of the calculations are presented in Appendix III.

It is worth mentioning that the tangent modulus values of Fig. 12 agree well with experimental data on 24S-T aluminium alloy in the range of the stress from 30,000 lb./in.^{2} to 40,000 lb./in.^{2} Similarly, the critical stress values in Fig. 13 are substantially correct for slenderness ratios between 40 and 60. For both shorter and longer columns the buckling stresses determined from equation (156) of Appendix III are too high.

To avoid complications in the analysis, the column was assumed to have an idealised I-section (*see* Fig. 14). This consists of two flanges spaced a distance h^* apart; each flange is assumed to have a cross-sectional area $A/2$. Many comparisons of theory and experiment have shown that this idealisation of the cross section does not introduce any significant errors in the analysis. The properties of the idealised cross section do not differ much from those of actual I-sections.

The static equation of equilibrium of the column was set up in Appendix III. Equation (168) is re-written here:

$$w''=-2\,(L/h^*)^2\,(1/2)^n\,[(p+m)^n-(p-m)^n] \quad (16)$$

This equation is valid for any n; in the numerical work n was taken as 3. The symbol w designates a non-dimensional displacement defined as

$$w=(y/\rho) \quad . \quad . \quad . \quad (17)$$

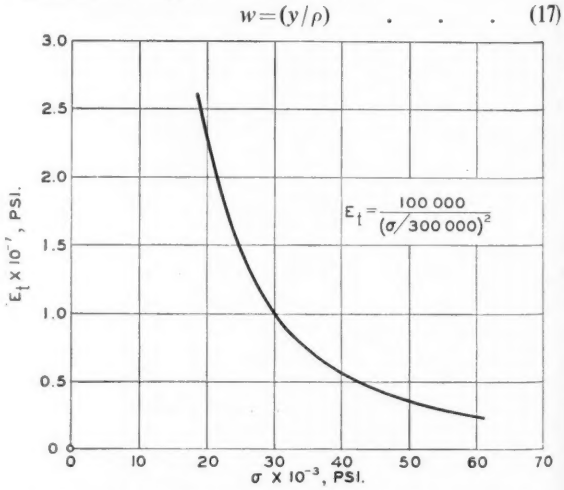


FIGURE 12. Tangent modulus corresponding to cubic law.

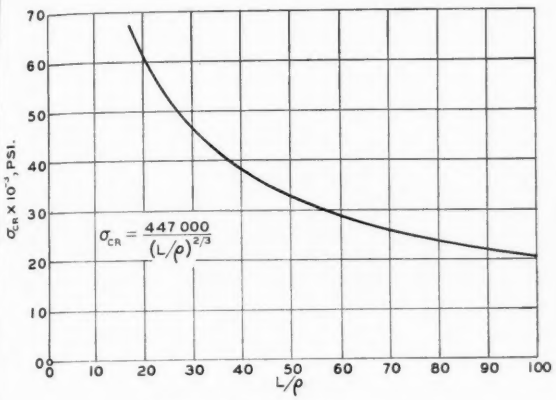


FIGURE 13. Column curve.

and y is the displacement shown in Fig. 3. The primes indicate differentiation with respect to x. The quantities p and m are also non-dimensional. They are defined as

$$p=(2P/AK) \quad \text{and} \quad m=(4M/AKh^*) \quad . \qquad (18)$$

and P is the axial force in the column (positive when tensile) and M the bending moment in the section of the column under consideration. The constant K is the one appearing in the stress-strain law, equation (14).

2.8. DEFLECTIONS OF THE COLUMN

When the column is initially perfectly straight, substitution of the equilibrium condition $m=-pw$ and of $n=3$ reduces equation (16) to

$$w''=2\Pi\,(3w+w^3) \quad . \quad . \quad . \qquad (19)$$

where

$$\Pi=\tfrac{1}{4}\,(L/h^*)^2\,p^3 \quad . \quad . \quad . \qquad (20)$$

Because of the cubic term, the assumption that $w=B\sin\pi x^*/L$ does not satisfy the equation, as the cube of the sine function is

$$\sin^3(\pi x^*/L)=\tfrac{3}{4}\sin(\pi x^*/L)-\tfrac{1}{4}\sin(3\pi x^*/L) \qquad (21)$$

The solution can be expressed, however, as an infinite trigonometric series and the coefficients of the series can be determined from conditions imposed upon

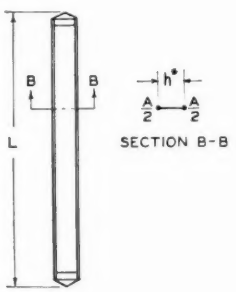


FIGURE 14. Column of idealised I-section.

the multipliers of like trigonometric terms. Such a calculation is made in Appendix III by means of a step-by-step approximation procedure. The connection between the applied (non-dimensional) load p and the amplitude a_1 of the first harmonic of the deflections is shown in the curve labelled $e=0$ in Fig. 15. Fig. 16 shows the corresponding amplitude a_3 of the third harmonic.

The contribution of the third harmonic to the deflected shape is seen to be negligibly small in this case. The second and fourth harmonics must vanish because of the symmetry; the higher harmonics are even smaller than the third, as may be judged from the derivations in Appendix III. Thus the calculations can be continued on the basis that in a good approximation the first harmonic alone is present.

This assumption was made when the deflections were calculated for a column whose axis, in its unstressed state, was slightly curved in the form of a half sine wave. The assumption is certainly a reasonable one in the light of what was found when the deflections of the initially straight column were investigated. It should be borne in mind, however, that the exact shape is not sinusoidal and that it keeps changing as the deflections increase. The investigation made here does not prove that the higher harmonics are negligibly small when n is greater than 3.

An equation similar to equation (19) is derived in Appendix III for the column with an initial sinusoidal deviation from the straight line (*see* equation (198)). The amplitudes a_1 calculated from this equation for various values of the eccentricity e are also plotted in Fig. 15.

The curves reveal the characteristic drop in the load carried by a non-linear column, after the maximum value is exceeded. They also show that the maximum depends greatly on the initial deviations from straightness. Similar observations were made by von Kármán in 1910 in connection with steel columns stressed beyond the elastic limit (*see* Reference 18).

2.9. THE DYNAMICS OF THE MOTION OF THE NON-LINEAR COLUMN

In analysing the dynamics of the motion of the non-linear column it is convenient to replace equation (4) by

$$(\partial^2 M/\partial x^{*2})+P\,(\partial^2 y/\partial x^{*2})-\mu A\,(\partial^2 y/\partial t^2)=0 \qquad (22)$$

where M is the bending moment in the section of

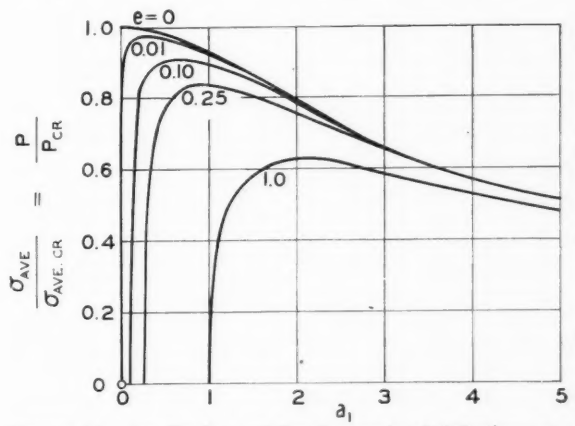


FIGURE 15. Amplitude a_1 of first harmonic of deflections as a function of compressive load.

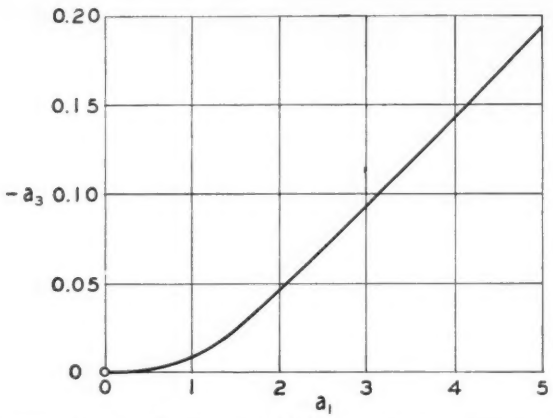


FIGURE 16. Amplitude a_3 of third harmonic as a function of amplitude a_1 of first harmonic for initially straight column.

the column under consideration. On the basis of the experience gained in the calculation of the static equilibrium, the initial deviations from straightness, as well as the deflected shape at all times, are assumed to be of the form of a half sine wave. All powers of the trigonometric functions are expressed in terms of the functions themselves and of the functions of multiples of their angles and, of each expression so obtained, only the first term is retained. In such a manner an approximate ordinary differential equation of the problem is derived which should be accurate enough for all practical purposes. The equation can be written in the form

$$(d^2a/d\xi^2)+\pi^2\Phi(ap+b)=0 \quad . \quad . \quad (23)$$

where a is the non-dimensional total displacement and b the non-dimensional moment at the middle of the column. The non-dimensional axial force p was defined in equation (18), and the non-dimensional time ξ in equation (6). Naturally a, b, and p are functions of the non-dimensional time ξ. The symbol Φ designates a new non-dimensional parameter, a dynamic similarity number, defined as

$$\Phi=K\varepsilon_E{}^2/2\mu c^2=(27/2)(E_{t,cr}/\mu c^2)\,\varepsilon_{cr}{}^{8/3} \quad . \quad (23a)$$

which is obviously related to Ω (*see* equation (9)).

The geometry of the deformations, together with the stress-strain relationship, yields one more condition connecting a, p, and b:

$$4p^2b+b^3=(32/3)\,\varepsilon_E(a-e). \quad . \quad . \quad (24)$$

Finally the displacement of the loading head is connected with the deformations of the column by the equation

$$\xi=(1/4)(a^2-e^2)-Sp-(1/16\varepsilon_E)(2p^3+3p^2). \quad (25)$$

In this last equation S depends on the elasticity of the testing machine. It was mentioned earlier that the snap process was possible only in an elastic testing machine. Naturally, every actual testing machine is elastic because its structural elements are made of elastic materials. Perfectly rigid testing machines exist only in theory. If this distributed elasticity is lumped together for ease of calculation and represented by a single linear spring with a constant C, as indicated in Fig. 17, the only place in the analysis where the effect of the elasticity of the testing machine appears is in equation (25). Moreover, the spring constant C enters the equations only in the following non-dimensional combination

$$S=(AK/2C)(1/L\varepsilon_E) \quad . \quad . \quad (26)$$

When the inertia term is disregarded in equation (23), the solution of the two simultaneous non-linear equations (23) and (24) is identical with the static solution discussed in the preceding section and illustrated in Fig. 15. To compute ξ from equation (25) for an elastic testing machine, the slenderness ratio of the column must be prescribed. The numerical results to be presented hereafter are all based on $L/\rho=50$. The connection between load and loading head displacement for a column whose slenderness ratio is 50 is shown in Fig. 18.

This diagram can be regarded as the history of the changes in the axial compressive force in the column

FIGURE 17. Column in elastic testing machine.

during a very slow loading process. When c is constant, ξ represents the time as well as the displacement of the loading head. Slow loading must be stipulated because the inertia forces were neglected in the analysis. The curves pertaining to the perfectly rigid testing machine ($S=0$) do not present any unusual features; they show a drop in load after the maximum value is reached, as can be expected with columns buckling above the elastic limit of their material. In Fig. 19 the variation in p is replotted to a different scale, and the variation in the strain on the concave and convex sides of the middle of the column is added. The entirely novel feature of Fig. 18 is that the curves corresponding to the elastic testing machine ($S=100$) are triple-valued in p over a considerable range of ξ.

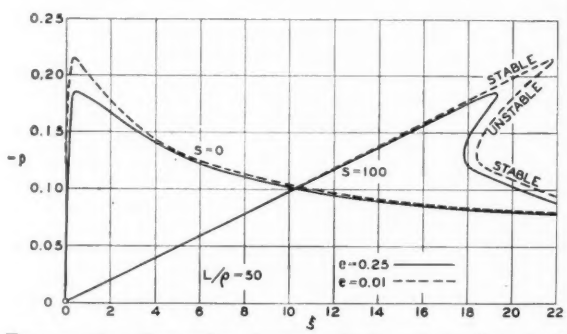


FIGURE 18. Connection between load and loading head displacement (static solution, non-linear column).

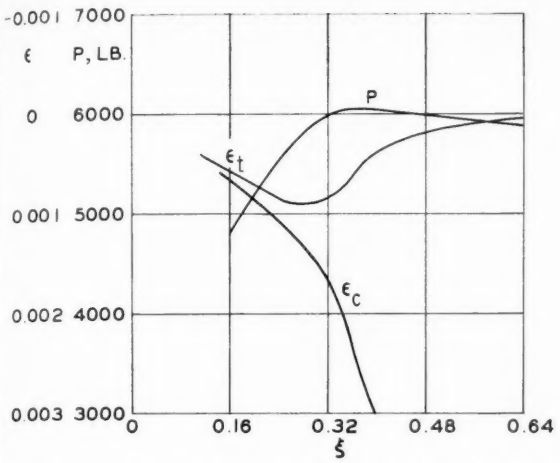


FIGURE 19. Load and strain calculated for short column in rigid testing machine.

This fact changes significantly the behaviour of the column in the testing machine. As the loading head descends, the load increases approximately linearly at the beginning of the test. The deviations from linearity increase when the maximum load is approached, but they are not large because the greater portion of the end displacement can be attributed to the elastic deformations of the testing machine, which take place in a linear fashion in accordance with the assumptions made.

Immediately after the maximum load is reached, the curve turns back; this reversed branch of the curve does not correspond to any real loading process because it can be followed only if the testing machine is reversed. If this is not done and the loading head continues to descend at a uniform rate, no points of equilibrium follow in a continuous manner. Equilibrium at further increased ξ values is possible only at significantly lower values of the load p which correspond to materially increased deflections a. Consequently, the column must jump from the small deflection position into the large deflection position, and during this snap-through process it must pass through positions in which static equilibrium is not possible. The resultant force in these positions does not vanish, but causes accelerations or decelerations in agreement with Newton's law.

A more detailed discussion of the properties of the equilibrium along the different branches of the curves labelled $S=100$ in Fig. 18 will be given later; first, the motion following a disturbance is investigated here.

This investigation can be made by means of the perturbation method and the asymptotic solution with respect to h^2 which were discussed at length in Appendix II. In the present case h^2 is equal to Φ rather than to Ω and the symbol χ stands for $\pi^2(ap+b)$. All the three functions a, b, and p must be perturbed; however, it is possible to express χ as a function of a alone, as was done in Appendix III. As a is a known function of ξ, χ can be plotted as a function of ξ.

The function χ corresponding to the non-linear column in the elastic testing machine differs greatly from that calculated for the perfectly elastic column. The first difference is that χ is very large when the value of a is close to e. This is a consequence of the stress-strain law adopted in the analysis. As mentioned earlier, equation (14) implies infinite rigidity at zero stress; correspondingly, the amplitude of any vibration introduced by a disturbance at a low stress must be very small. Theory is in agreement with this conclusion because the amplitude of the vibrations is proportional to $\chi^{-\frac{1}{4}}$ in the asymptotic solution (*see* equation (95)). The asymptotic solution developed here has little practical significance at low stresses because the properties of actual materials differ considerably from those assumed in the low stress range.

At the higher stresses prevailing at the time when the column buckles, the stress-strain relationship assumed is in good agreement with reality. There the asymptotic solution should therefore predict the vibrations of the column in a reliable manner. But in this range χ again differs significantly from the function obtained for the perfectly elastic column. With the latter, χ was found to be always positive and to have a minimum just before the buckling load was reached. With the non-linear column χ can be negative, as is shown in Fig. 20.

As a matter of fact, χ is negative for the unstable branch of the equilibrium curve in Fig. 18, which does not correspond to the routine process of loading, as was discussed earlier. It is positive for the top and bottom branches of the curve along which the equilibrium is stable. When χ is positive, the motion following a disturbance consists of vibrations about the static equilibrium positions according to equation (96). When χ is negative, the solution can be expressed by a hyperbolic function and the deflections increase monotonically. Soon they become so large that the perturbation solution loses its validity.

Two conclusions can be drawn from the investigation. First, the earlier conjecture concerning the stability of the various branches of the curves of Fig. 18 is confirmed. A small disturbance results only in small vibrations about the positions of the column represented by the upper and lower branches of the curve; along those curves the equilibrium is therefore stable. When the equilibrium corresponding to the intermediate branch is disturbed, the deflections increase without limit and thus the equilibrium is unstable.

The second conclusion concerns the growth of the amplitude of the vibrations following a disturbance of the equilibrium represented by the upper branch of the curve. As χ decreases with increasing compressive load and vanishes when the buckling load is reached, it follows from equation (96) that the magnification of the amplitude of the vibrations increases greatly as the buckling load is approached. Thus a small disturbance leads to large displacements when the compressive load is slightly smaller than the buckling load.

The motion of the column following such a disturbance will now be analysed further, with the aid of diagrams drawn in the so-called phase plane.

2.10. DIAGRAMS IN THE PHASE PLANE

It is now assumed that the loading head of the testing machine is lowered to some position ξ and then fixed in

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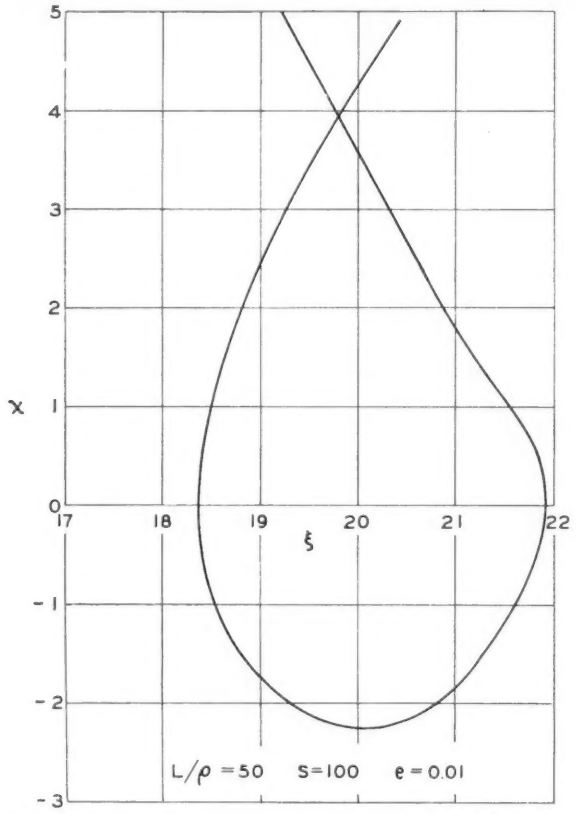


FIGURE 20. Function $\chi(\xi)$ for short column in elastic testing machine.

that position. In this section ξ is therefore the non-dimensional measure of a fixed position of the loading head, and it no longer represents time. To take care of this new viewpoint, equation (23) is multiplied by $(c/L\varepsilon_E)^2$; if the definition of Φ given in equation (23*a*) is taken into account, the resulting equation can be written as

$$(d^2a/dt^2)+K^*(ap+b)=0 \quad . \quad . \quad (27)$$

with $$K^*=(\pi^2/2)(K/\mu L^2). \quad . \quad . \quad (28)$$

The functions a, p, and b are still interconnected because of equations (24) and (25), but ξ in the latter has now a prescribed value. Because of this inter-relationship the quantity $(ap+b)$ can be expressed as a function of the variable a and the parameter ξ. If the lateral velocity v is introduced as

$$v=da/dt \quad . \quad . \quad . \quad (29)$$

multiplication of Equation (27) by $da=vdt$ results in

$$vdv+K^*f(a)\,da=0 \quad . \quad . \quad (30)$$

where $$f(a)=ap+b. \quad . \quad . \quad . \quad (31)$$

After integration this becomes

$$(1/2)v^2+K^*\int_{a_1}^{a} f(a)\,da+C'=0. \quad . \quad (32)$$

The integral can be evaluated numerically; if it is designated by I^*

$$I^*=\int_{a_1}^{a_2} f(a)\,da. \quad . \quad . \quad . \quad (33)$$

It follows then from Equation (32) that

$$v=K^{**}(C_1-I^*)^{\frac{1}{2}} \quad . \quad . \quad . \quad (34)$$

where $$K^{**}=(2K^*)^{\frac{1}{2}}=(\pi^2K/\mu L^2)^{\frac{1}{2}}. \quad . \quad . \quad (35)$$

If the deflection a_1 is now specified as the lower limit of the integral, and if a value is selected for the constant C_1, equation (34) yields two equal and opposite velocities v for every value of the displacement a. Corresponding values of a and v can then be plotted in a Cartesian plane; a different curve is obtained for each choice of C_1. The $a-v$ plane is known as the phase plane. Its use in vibration analysis is discussed at length and in an easily understandable manner by James J. Stoker in his book "Nonlinear Vibrations in Mechanical and Electrical Systems"[25]. The problem of the dynamic stability of a simplified model of a column is also treated in the book on p. 54.

In computing the numerical values of the function $f(a)$ it is convenient to introduce the notation

$$k^2=-(4/3)(\varepsilon_E/p) \qquad R^2=4\xi+e^2+4Sp+(p^3/2\varepsilon_E)$$
$$A=(a/R) \qquad B=(b/kR). \quad . \quad . \quad (36)$$

With these symbols Equations (24) and (25) become

$$A^2+B^2=1 \quad . \quad . \quad . \quad (37)$$

$$A=(e/R)+(3/8\varepsilon_E)kp^2B+(3/32\varepsilon_E)k^3R^2B^3. \quad (38)$$

Equation (37) represents a unit circle in the $A-B$ plane. The quantities e, ε_E, S, and ξ are known and thus k and R can be computed. The values of A and B, and consequently those of a and b, correspond to the points of intersection of the parabola of equation (38) with the unit circle of Equation (37). The intersection points can be obtained graphically or by a numerical trial and error process. In all the computations performed k^2 and R^2 were positive, the parabola represented a single valued function B of A, and two real intersection points existed.

Figure 21 is the phase plane diagram corresponding to $\xi=18$. Each closed curve in the figure represents a possible sequence of states of the motion of the column. The highest velocity occurs when the column moves through the points of static equilibrium; the velocity is positive in the upper half plane and negative in the lower half plane. As the midpoint of the column moves to the right from the equilibrium position, the velocity decreases and finally becomes zero. In this position, of course, the restoring force has its maximum and it causes the column to swing back towards its static equilibrium position.

It is of interest to observe that the various curves shown are not similar and thus the velocity-displacement relationship depends upon the magnitude of the amplitude. Even the period of the vibrations depends upon the amplitude as a further integration would show.

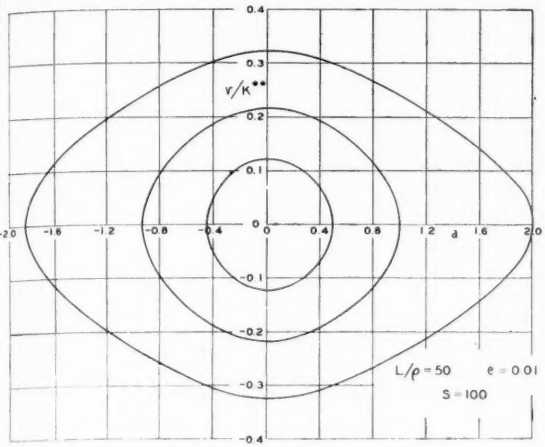


FIGURE 21. Phase plane diagram for $\xi = 18$.

These two facts are consequences of the non-linearity of the system. A more detailed theoretical and experimental investigation of the natural modes of vibration of a column was made in a thesis submitted to the Polytechnic Institute of Brooklyn by David Burgreen in 1950[26].

The diagram representing the column in the phase plane is entirely different when $\xi = 19$, as can be seen from Fig. 22. A small disturbance from the static equilibrium position again results in oscillations about the equilibrium position, but sufficiently large disturbances cause the column to snap through into either of the large deflection equilibrium positions to the right or the left. Each stable equilibrium position is at the bottom of a valley if Fig. 22 is regarded as a topographical map. A somewhat smaller transverse displacement, or initial velocity of disturbance, suffices for the column to snap to the right than is necessary for it to snap to the left. This bias is a consequence of the small initial deviation from straightness.

Between any two stable equilibrium positions there is a saddle point on the map which represents unstable equilibrium. The saddles correspond to the middle branch of the curve in Fig. 18, while the lowest points of the valleys correspond to the points along the upper and lower branches of the curve $S = 100$, $e = 0{\cdot}01$.

One advantage of the phase-plane approach over the perturbation approach should now be clear; the latter is only valid for small disturbances while the former can represent the effect of disturbances of any magnitude. In both approaches the internal friction, as well as the friction along the knife edges, is disregarded. The effect of friction would appear in the phase plane in a distortion of the paths; in the presence of friction the motion of the column is characterised by paths which spiral into the stable equilibrium positions as energy is lost during the motion. The disadvantage of the phase-plane approach is that it can be used only when ξ is a constant; otherwise equation (32) cannot be integrated in the simple manner shown.

Figure 22 illustrates the interesting fact that the equilibrium of the column in the small-deflection position is stable when the disturbance is small and unstable when it is large. The figure also contains quantitative information relative to the minimum value of the disturbance velocity or displacement that suffices to start the snap process.

Finally, in Fig. 23 the conditions very close to the buckling load, namely at $\xi = 21{\cdot}89$, are presented. The diagram is essentially a degenerate case of Fig. 22, inasmuch as the two saddle points almost coincide with the static equilibrium point of small deflections. The stable small-deflection equilibrium position is so close to the unstable equilibrium position corresponding to a positive displacement that it is impossible to show it separately in the figure. While at $\xi = 19$ a deflection of almost 3ρ (that is, $a = 3$) is required before the column can leave the stable valley at the middle of the figure, at $\xi = 21{\cdot}89$ the distance is so small that it cannot be shown. A disturbance of this magnitude is easily possible in this region because χ is very small; thus the magnification of any disturbance is very substantial, as was shown earlier.

The only remaining phase of the buckling process that still has to be calculated is the snap through proper. This can be done without difficulty on the basis of Fig. 23. Let it be assumed that some disturbance carries the

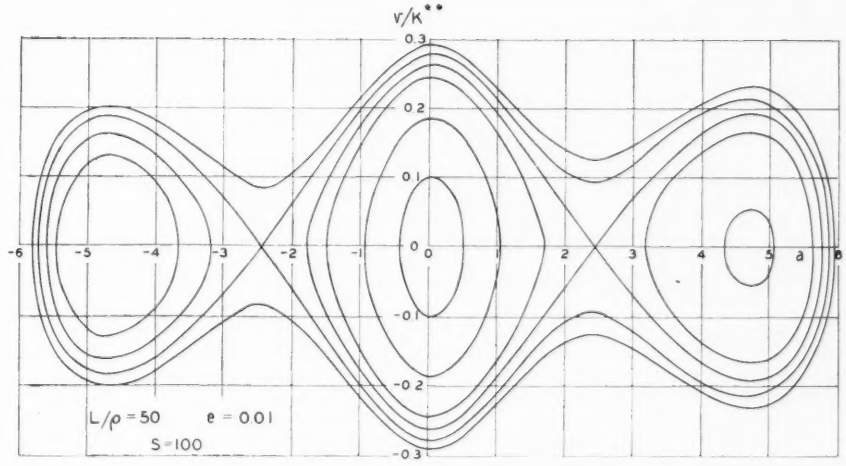


FIGURE 22. Phase plane diagram for $\xi = 19$.

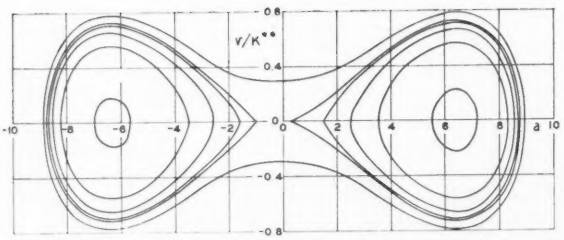


FIGURE 23. Phase plane diagram for $\xi = 21{\cdot}89$.
$L/\rho = 50 \quad e = 0{\cdot}01 \quad S = 100$

column into the state characterised by $a = 0{\cdot}5$, $v/K^{**} = 0{\cdot}02$ in the diagram drawn for $\xi = 21{\cdot}89$. Subsequent states are then characterised by points on the curve passing through this starting point. Corresponding values of a and v/K^{**} can easily be read from this curve at a number of stations and values of K^{**}/v can be plotted against a. But solution of the equation defining v, namely equation (29),

$$v = da/dt$$

yields the time necessary to cover the distance $a_2 - a_1$:

$$t_2 - t_1 = \int_{a_1}^{a_2} (1/v)\, da. \qquad . \quad . \quad (39)$$

Integration of the K^{**}/v function with respect to a yields therefore the time sought, provided that the integral is divided by K^{**}. The numerical work was done for $\rho = 0{\cdot}108$ in., $E_{t,cr} = (1/3) K/\varepsilon_{cr}^{2/3} = 8{\cdot}34 \times 10^6$ lb./in.^{2}, and thus for

$$(E_{t,cr}/\mu)^{\frac{1}{2}} = 179{,}000 \text{ in./sec.}$$

With these values one obtains

$$K^{**} = 19{,}700. \qquad . \quad . \quad . \quad (40)$$

The time necessary for the column to cover the distance from the starting point to the position of equilibrium at large deflections was found to be 0·0015 sec. The theory therefore predicts the snap-through process.

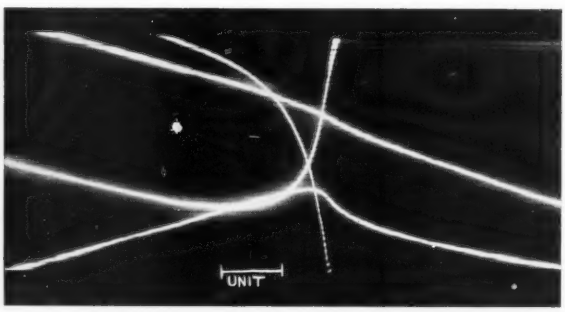


FIGURE 24. Test record for short column in rigid testing machine ($L/\rho = 50$). The unit length shown corresponds approximately to 1 sec., 360×10^{-6} strain, 0·00175 in. head displacement, and 900 lb. load. The initial values of the strains are equal. They are projected apart for the sake of clarity.

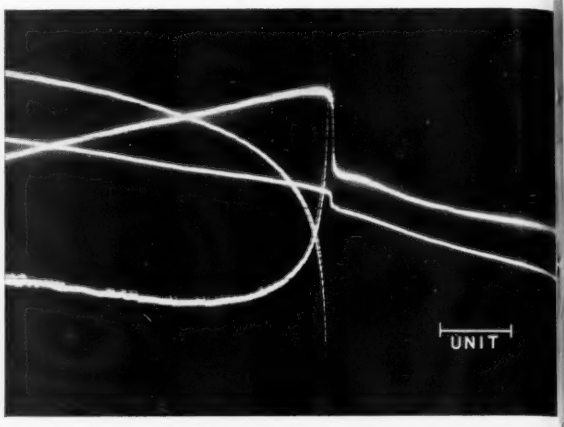


FIGURE 25. Test record of short column in semi-rigid testing machine ($L/\rho = 50$). The unit length shown corresponds approximately to 1 sec., 360×10^{-6} strain, 0·00375 in. head displacement, and 900 lb. load. The initial values of the strains are equal. They are projected apart for clarity.

2.11. EXPERIMENTS ON THE ELASTICITY OF THE TESTING MACHINE

As the theory had shown that the snap action was a consequence of the elasticity of the testing machine, it appeared desirable to prove this conclusion experimentally also. Fortunately a 200,000 lb. Riehlé screw-type testing machine turned out to be rigid enough for the purpose.

Figure 24 is the oscillograph record of the test of

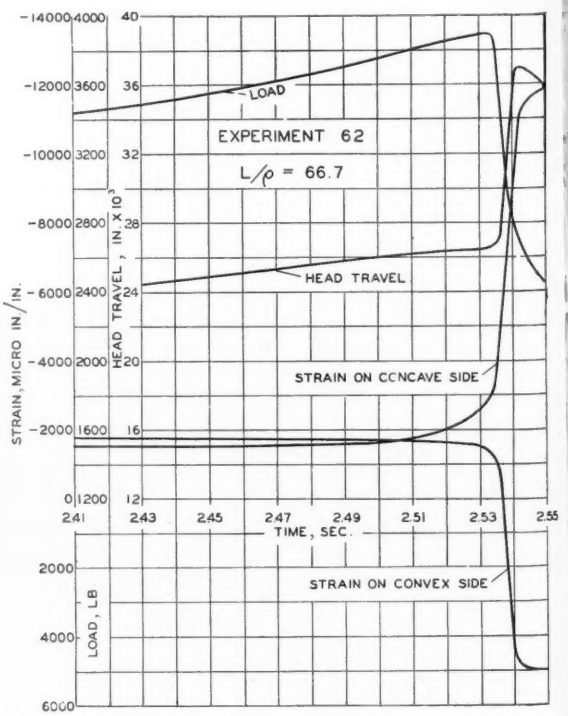


FIGURE 26. Results of buckling test with medium length column in hydraulic machine.

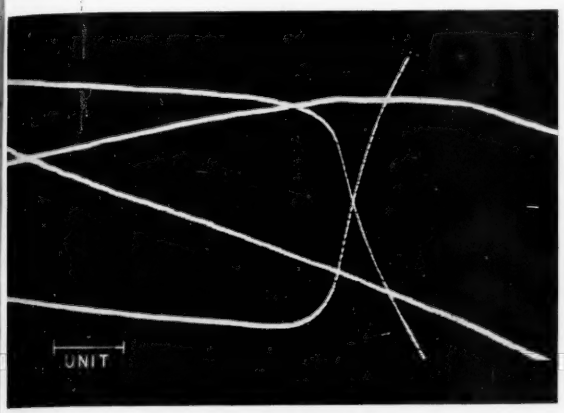


FIGURE 27. Record of medium-length column in rigid testing machine ($L/\rho = 66{\cdot}6$). The unit length shown corresponds approximately to 1 sec., 360×10^{-6} strain, 0·00175 in. head displacement, and 900 lb. load. The initial values of the strains are equal. They are projected apart for the sake of clarity.

a column in the Riehlé machine. The column had a slenderness ratio of 50 and it supported a maximum load of 6,400 lb. The traces are from the top to the bottom, as they appear to the left of the maximum load: Strain on concave side, loading head displacement, strain on convex side, and compressive load. It can be seen that the rate of descent increased slightly at the moment of buckling but a sudden drop, as in Fig. 9, did not develop. There is no jump visible in any of the traces.

Experiments were also made to determine the elasticity of the testing machine. As the buckling tests were run with the rider at zero load and the weighing arm pressed against its stop, the elasticity had also to be measured under the same conditions. An automobile jack was placed in the machine and a load amounting to about half the buckling load of the column was applied to the jack by operating the loading head. Subsequently the jack was raised until the buckling load was reached. The load was measured by a rather rigid strain gauge dynamometer placed between the jack and the lower platen. Next the load in the jack was slowly released by operating the jack and simultaneously readings were made of the load and of the distance between the two platens. The latter was measured by means of Ames' dial gauges. Finally a correction was made for the elastic deformations of the load dynamometer.

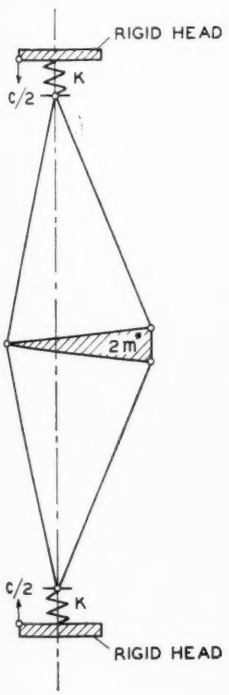


FIGURE 28. Model column in an elastic testing machine.

When the load was plotted against the changes in distance, the slope of the straight line obtained gave a spring constant of about 10^6 lb./in. Another test was run on a column with a more elastic dynamometer between it and the lower platen. The cross-sectional area of this thin-walled cylindrical dynamometer was 0·4 in.^{2} and its length was 2 in. Under a load of 1,000 lb. this aluminium alloy cylinder foreshortened 5×10^{-4} in; hence its spring constant was 2×10^6 lb./in. The combination of the Riehlé machine with this dynamometer had an overall spring constant of $6{\cdot}66 \times 10^5$ lb./in.

The record of a test made in the Riehlé machine with the elastic dynamometer is reproduced as Fig. 25. The slight increase in the elasticity sufficed to cause a jump. At buckling the load dropped suddenly close to 1,000 lb. and the loading head was displaced about 10^{-3} in. The drop was accompanied by an audible thud. The buckling load was again 6,400 lb.

Finally calibration of the Sonntag hydraulic machine gave a spring constant of 150,000 lb./in. The high elasticity of this machine was the cause of the explosive nature of all the buckling processes recorded in the first series of tests. An example of this was given in Fig. 9. The record of another test in the hydraulic machine was evaluated and the results were plotted in Fig. 26. It can be seen that the snap-through process was violent; the load dropped 1,300 lb. in 0·01 sec.

Figure 26 should be compared with Fig. 27 because both represent columns of medium length ($L/\rho = 66{\cdot}7$). The former was tested in the elastic hydraulic machine and the latter in the rigid screw machine. In the rigid machine the maximum load was maintained practically

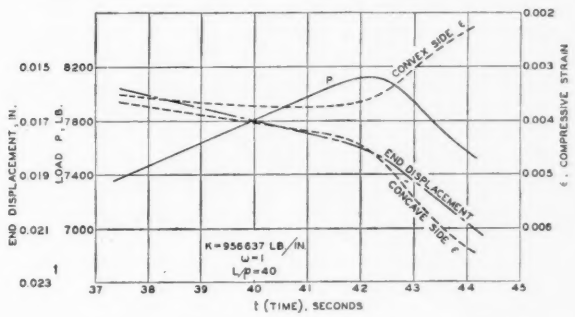


FIGURE 29. History of buckling of model column.

constant for a while, and then the load started to decrease gradually. In the elastic machine the column snapped through.

2.12. MODEL OF A COLUMN

Comparison of the theoretical curves of Fig. 19 with the test results shown in Fig. 24 shows general agreement, but also many differences in detail. This is not surprising if it is remembered that the theory is based on a highly simplified stress-strain law. In fact, it is most gratifying that the far-reaching idealisation of the stress-strain relationship does not lead to greater discrepancies between theory and experiment.

Nevertheless it was considered desirable to develop a theory based on a more realistic stress-strain law. To make this possible, a simplified model of a column was developed, as shown in Fig. 28. All the results related to this model are taken from a thesis submitted to the Polytechnic Institute of Brooklyn by Frank J. Romano in 1953[27].

The model column consists of four pin-ended weightless bars attached to a rigid wedge of mass $2m^*$. The equation governing the lateral motion of this mass can be made to agree essentially with equation (75) if the material of the bars is perfectly elastic. The governing equations become more complex when the material properties are taken from the results of compression tests made with actual materials, but the integration of the equations is still possible, at least by means of numerical methods.

Figure 29 presents the results obtained with a 24S-T aluminium alloy column of a slenderness ratio of 40. The effect of the strain reversal was taken into account in the analysis by using Young's modulus in the calculation of the stress when the strain was decreasing. Similarly, the effect of the elastic deformations of the testing machine was included by assuming for the machine a spring constant of 956,637 lb./in. (corresponding to Romano's parameter $\omega=1$). The shapes of the curves resemble the traces in Fig. 24.

2.13. MEASUREMENT OF INITIAL DEVIATIONS FROM STRAIGHTNESS

As the initial deviations of the column from straightness play an important role during the buckling process, an effort was made to determine the initial shape of well-centred columns[28]. Six columns having slenderness ratios ranging from 51 to 112 were analysed. The centring was carried out under load in the usual manner with the aid of strain gauges in the middle of the column. The test set-up is shown in Fig. 30. The longest columns were provided with 25 pairs of strain gauges, and the changes in the curvature of the columns at the location of the gauges were recorded under a number of loads. These changes were subjected to a harmonic analysis and the Fourier components of the initial shape were calculated from the elastic theory of beam-columns. Consistent results were obtained in all cases.

The order of magnitude of the amplitude of the first Fourier component of the initial deviations was found to be 10^{-4} in.; this corresponds to $e=10^{-3}$. The second component was 3 to 12 times larger.

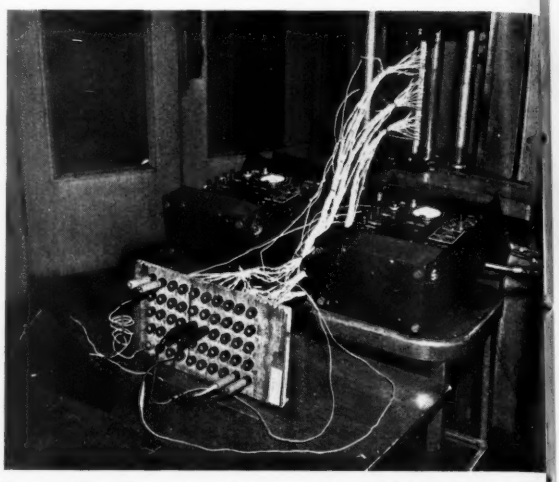


FIGURE 30. Test set-up for measuring initial deviations from straightness.

2.14. TH. VON KÁRMÁN'S OBSERVATIONS

Probably few persons know today that the snap-through process of short columns was observed by Th. von Kármán in his work for the doctor's degree and published in his thesis in 1910[18]. Fig. 31 is reproduced here from this thesis (there it was Fig. 29). Dr. von Kármán's remarks on the buckling process of columns of medium slenderness ratio are reprinted in Appendix 1*g* and a translation is given here:

"After the maximum load was reached, most columns buckled so suddenly that the manometer and

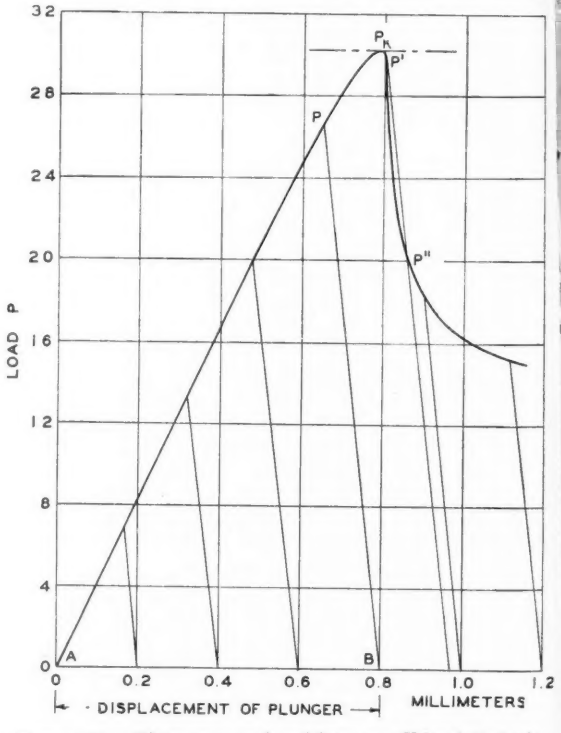


FIGURE 31. Diagram reproduced from von Kármán's thesis.

the deflection gauge received a strong jolt. . . . The explanation is that the equilibrium of the composite system consisting of the pump, the testing machine, and the column becomes unstable. To clarify the process, one should consider the load history in the machine as a function of the decrease in the distance between the two knife edges. The latter is equal to the displacement of the plunger less the elastic distortion of the testing machine which in turn is composed essentially of the stretching of the support columns of the machine and the deflection of its cross beam. The elastic distortion can be assumed to be proportional to the load acting upon the test column. In Fig. 31 the plunger displacement is plotted along the x-axis, and the force along the y-axis; the state of the machine corresponding to any given amount of fluid in the cylinder is then represented by an inclined straight line such as BP. On the other hand decrements in the length of the test column correspond to a load history which is represented by curve AP_k as a function of the decrements. The points of intersection of this curve with the set of parallel straight lines determine the sequence of states of equilibrium of the composite system. As the curve drops rapidly after the maximum load P_k is passed, a point P' is reached where one of the straight lines is the tangent of the curve. At this point a sudden change in the state of equilibrium takes place as the load diminishes rapidly until at point P'' a stable state of equilibrium is again established."

3. Buckling under Various End Conditions

3.1. THE EFFECT OF THE END CONDITIONS

In classical column theory, end conditions are discussed in some detail and distinction is made among rigidly fixed, simply supported, and free ends. As far as the load is concerned, it is customary to assume that it acts in the vertical downward direction and is independent of the elastic deformations of the column. This traditional definition of the load is based on the field concept which has had an overwhelming influence upon mechanics (*see* Professor Ziegler's paper[29]). When buckling loads derived from this theory are checked in experiment, it is often overlooked that the actual loads applied do not have the properties stipulated in the theory.

There are few problems in experimental mechanics which cause more difficulty than the application of a truly constant load to a column. When the load is applied in an ordinary testing machine, the end deflection rather than the load is prescribed, as was discussed in the preceding Section. The oscillographic records of tests with short columns have shown that the load does not remain constant, but drops very rapidly during the buckling process. A similar situation exists when closed thin-walled cylinders are subjected to a hydrostatic pressure in a tank. The rapidity of the drop in load at buckling depends on the elasticity of the tank, the pump, and the piping.

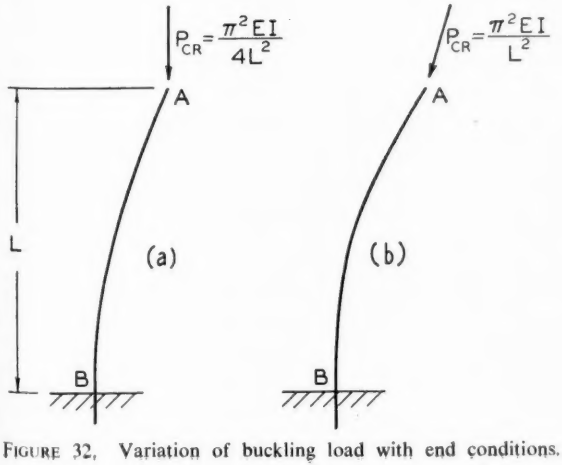


FIGURE 32. Variation of buckling load with end conditions.

Loads not only change in magnitude but are also likely to change in direction in consequence of the deformations of the test specimen. This is not true of the pin-ended column in the ordinary testing machine, but it is an ever-present problem in load applications by means of levers, jacks, and linkages. It is easy to show that changes in the direction of the load have a profound effect upon the magnitude of the buckling load.

Figure 32(a) represents the conditions originally investigated by Euler. The load remains vertical during buckling and the critical load is $P_{cr}=\pi^2 EI/4L^2$. If the applied load is required to change its direction in such a manner that it always passes through point B (as well as through point A) in the figure, obviously its moment arm must always remain zero at both ends of the bar. But the shortest segment of a sine curve which satisfies these conditions is the half sine wave. Thus the column of Fig. 32(b) buckles according to the half-sine wave pattern just as the column whose two ends are pinned; this implies that the buckling load is $P_{cr}=\pi^2 EI/L^2$. The change in the direction of the load, which can remain unnoticed in an experiment, increases the buckling load by a factor of 4.

Another possible relationship between load and deformation is expressed by the requirement that the load always remain tangential to the free end of the column. It has been concluded from static considerations that under these conditions the column cannot buckle at all (*see* Reference 30). Recently Max Beck showed that a dynamic criterion yields $P_{cr}=20\,EI/L^2$ as the stability limit under these conditions[31].

3.2. TSIEN'S STUDY OF THE INFLUENCE OF THE TESTING MACHINE

The end conditions are imposed upon the test specimen by the testing machine. Nevertheless comparatively little attention has been paid by investigators to the interaction between the column and the testing machine, a notable exception being Th. von Kármán[18]. In a remarkable paper published in 1942[32] H. S. Tsien analysed the buckling of non-linear structural elements and the influence of the testing machine upon their buckling. The study was made to verify rigorously some of the conclusions reached in an analysis of shell buckling carried out by means of the energy method.* The equilibrium of thin shells is governed by simultaneous partial differential equations; they are so complex that a solution by other than energy methods is hardly possible. Professor Tsien was able to reproduce the essential non-linear characteristics of the thin shell

* This work is discussed further in Section 6.

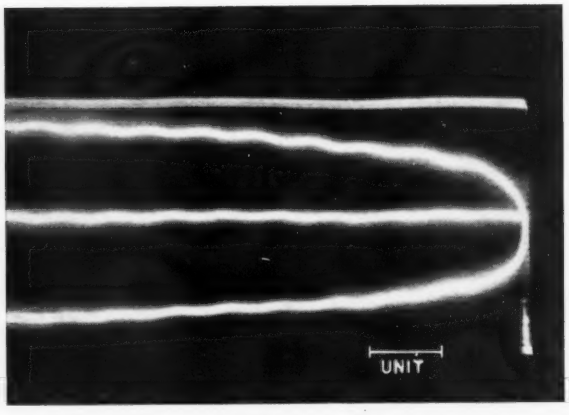


FIGURE 33. Record of collapse of intermediate column in dead-weight test ($L/\rho = 66{\cdot}6$). The unit length shown corresponds approximately to 1 sec., 360×10^{-6} strain, 0·04 in. head displacement, and 180 lb. load. The initial values of the strains are equal. They are projected apart for the sake of clarity.

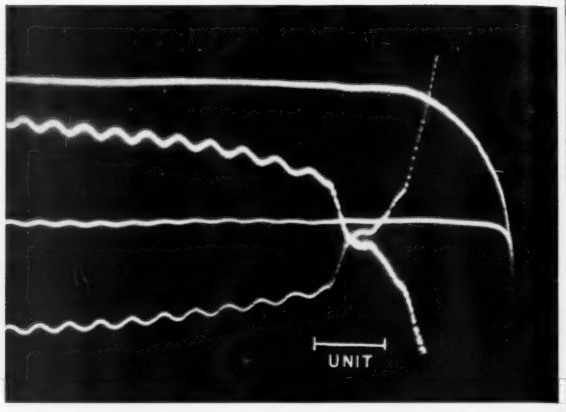


FIGURE 34. Record of collapse of long column in dead-weight test ($L/\rho = 112$). The unit length shown corresponds approximately to 1 sec., 360×10^{-6} strain, 0·04 in. head displacement, and 180 lb. load. The initial values of the strains are equal. They are projected apart for the sake of clarity.

with a model consisting of an ordinary pin-jointed linear column supported in the transverse direction by non-linear springs. In the paper a complete solution of the buckling problem of such a column was obtained rigorously in closed form.

The main findings of the investigation are that three equilibrium states can correspond to a single value of the displacement of the loading head of the testing machine when this displacement is sufficiently large. The triple-valued range is increased by the elasticity of the testing machine and decreased by initial deviations of the column from the straight line. When the initial deviations are large enough, the triple-valued range does not exist.

In a plot of the load against the loading head displacement, the middle portion of the curve, lying between its two vertical tangents, is unstable. A comparatively small disturbance suffices to cause the column to jump from the small-deflection stable state of equilibrium through the intermediate unstable state into the final state of large-deflection equilibrium. As the load in the latter state is smaller than in the initial stable state corresponding to small deflections, a small disturbance may lead to buckling at a load smaller than the critical load of the small deflection theory.

The analysis was made on the basis of considerations of static equilibrium. For this reason an assumption had to be made for the conditions under which the jump could take place. Professor Tsien suggested that the jump was most likely to occur at that loading head displacement at which the elastic energy stored in the system was the same in the small deflection state as in the large deflection state.

A related study dealing with the linear elastic column supported by a non-linear spring was published by H. L. Cox in 1940[33].

3.3. DEAD WEIGHT LOADING

It is generally assumed in engineering that loading by dead weights fulfils the theoretical requirements as to the constancy of the load. The assumption is certainly correct as far as the direction is concerned. The weight of the mass applied in a dead-weight test programme also remains unchanged, but the force transmitted from the mass to the specimen changes rapidly in consequence of the inertia effects. When the column buckles, it takes more time to accelerate the weight in the downward direction than to obtain a relief in the column load because of the lateral deflections.

These conditions were observed in recent tests made at the Polytechnic Institute of Brooklyn. The axial compressive load was transmitted to the column through a lever. The ends of the column were provided with knife edges. From the end of the lever a container was suspended into which water was flowing from the city water system during the test. The displacement of the upper knife edge relative to the lower knife edge, the compressive load in the column, and the strains on the convex and concave sides of the middle of the column were measured by bonded wire strain gauges and were recorded in oscillograms.

Figure 33 shows the collapse of a column having a slenderness ratio of 66·6. The column snapped through in a manner similar to that observed in testing machine loading. The process of buckling of a long column is illustrated in Fig. 34 ($L/\rho = 112$). The load remained sensibly constant for a few seconds after its maximum value was reached in the test. During this time the lateral deflections of the column increased steadily, as can be concluded from the increase in bending strain visible in the figure. When the strain exceeded the elastic limit of the material in a sufficiently large portion of the column, its resistance to flexure began to decrease and the column collapsed suddenly, as can be seen from the rapid drop in the values of the load and the end displacement. The waviness of the strain traces is a consequence of vibrations induced by variations in the water pressure.

These tests indicate that dead-weight loading does not provide a constant load in the buckling test. A very

soft spring, such as a large container of compressed air, incorporated in the testing machine is most likely to have the desired effect.

3.4. RAPID LOAD APPLICATION

Practical experience has shown that columns can support axial compressive loads considerably in excess of the critical loads if the loads are applied for only a short time. If this were not true, it would hardly be possible to drive a nail with a hammer.

The effect of rapid load application on perfectly elastic columns was investigated by the author[22] and by the author jointly with S. V. Nardo and Burton Erickson[34]. It was assumed that the displacement of the upper end of the column relative to its lower end, rather than the force itself, was prescribed in the test. Under these conditions the differential equations of motion derived earlier are still valid, provided that the load is not a true impact load; it must not be applied with a rapidity comparable to the velocity of the propagation of sound in the material of the column. In a true impact loading equations (69), (70) and (71) must be solved simultaneously. Tests have shown[35] that under such conditions the column can buckle with a wave length much shorter than the column length.

In the rapid, but not impact-type, loading under consideration equations (4) and (5) apply; this does not indicate, however, that the method of solution presented in Appendix II can again be used. The type of solution selected must depend upon the value of the parameter Ω. In a rapid loading Ω is small and the static solution of the problem ($\partial^2 y/\partial t^2 = 0$) does not represent a first approximation to the dynamic solution. The dynamic deflections are much smaller than the static ones at the beginning of the loading process because it takes time for the column to deflect laterally.

This indicates that all deflections must be small at the beginning of a rapid test if the initial deviations from straightness are small. For this reason the equation of motion, as presented in equation (75), can be linearised. The linearised equation is of the Bessel type and can be solved, with the appropriate boundary conditions, as is shown in Reference 22.

Deflections increasing monotonically with time are obtained upon which rapid oscillations of small amplitude are superimposed. This type of motion changes over to another type when the Euler displacement $\xi_E = 1 = ct/L\varepsilon_E$ is exceeded; beyond the Euler displacement the solution takes the form of modified Bessel functions which do not oscillate.

Depending upon the speed of load application and the initial deviations from straightness, the deflections sooner or later become too large for the linearised equation to remain valid. Then recourse must be had to a series expansion; this can happen above the Euler displacement, and may even occur below it. When the deflections are very large, a perturbation solution based on the static solution again becomes a method suitable for the solution of the buckling problem. In addition to the methods enumerated, a numerical integration of the differential equation is also feasible in all stages of the loading.

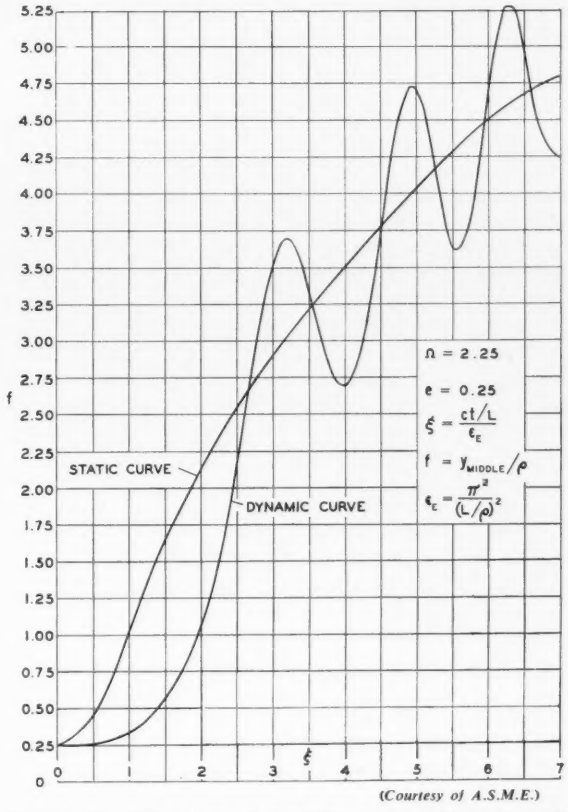


FIGURE 35. Deflections of perfectly elastic column in rapid loading test.

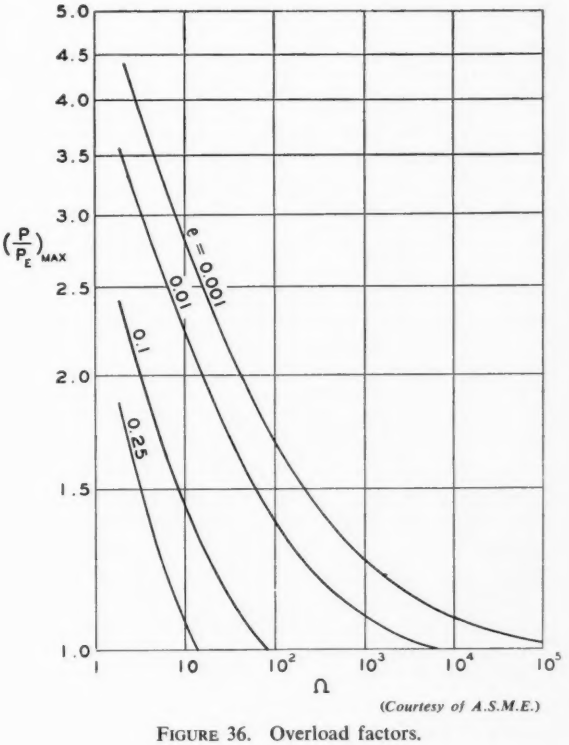


FIGURE 36. Overload factors.

Figure 35, taken from Reference 22, shows how the dynamic deflection lags behind the static deflection in the initial stages of the loading process. Soon the dynamic deflection over-shoots the static value after which oscillations of large amplitude follow. The figure represents the conditions as they prevail in very rapid loading. The value $\Omega = 2{\cdot}25$ of the dynamic similarity number corresponds to a loading head velocity of about 4 in./sec. when the slenderness ratio of the column is 150. Curves illustrating the deflection and load history of columns were given for various values of Ω and e by J. P. Chawla in a thesis submitted to the Polytechnic Institute of Brooklyn[36].

Naturally, the load in the column increases proportionately to the loading head displacement when there are no lateral deflections. As in a rapid test the deflections are small, the compressive load in the column can over-shoot the Euler load considerably, so long as the material remains perfectly elastic. The over-load factor, defined as the ratio of the maximum load attained in the column test to the Euler load, is of considerable practical interest with structural elements subjected to rapidly applied loads. Values of the over-load factor calculated by Hoff, Nardo, and Erickson are reproduced from Reference 34 as Fig. 36. They are considerably greater than unity when Ω and e are small.

3.5. RAPID LOADING TESTS

As it was considered desirable to check experimentally the predictions of the theory, a rapid loading machine for columns has been designed and constructed at the Polytechnic Institute of Brooklyn. Fig. 37 is a photograph of the machine. It is essentially a flywheel with an interchangeable cam which actuates a follower to depress rapidly the upper end of a column. The flywheel is brought up to full speed by a squirrel cage induction motor rated at 2 h.p. at 1,680 r.p.m. The V-belt drive is arranged in such a manner that the maximum rotational speed of the flywheel is 2,000 r.p.m.

The flywheel is mounted on a shaft of 3 in. diameter which runs on ball bearings housed in heavy steel parts. A standard hydraulic automobile brake is provided to stop the flywheel and a tachometer indicates the revolutions per minute of the shaft.

The machine accommodates columns of a maximum length of 13 in. The column visible in the photograph is provided with knife edges. The lower knife edge rests on a bearing plate which can be adjusted in the vertical direction by a base screw of 2 in. diameter provided with a square thread. This screw is used to load the column when it is centred in the preliminary static tests.

When the flywheel is rotating at the desired speed, the operator throws a switch which actuates a trigger mechanism. The trigger brings the follower in position to engage the cam; a timing commutator is provided to ensure that this can happen only when sufficient time is available for the follower to become fully operative.

Since the rapid loading machine is still in the adjustment and calibration stage, only a few preliminary tests have been made in it. Fig. 38 is the oscillographic record of such a test. The two strain gauges bonded to opposite sides of the top of the column were connected in series, as were also the two at the bottom; the combined signal of the two pairs, which in turn were again connected in series, was used to indicate the average strain in the column. As the test was run entirely in the elastic range, the average strain was a measure of the load in the column. This load is represented by the trace which is at the bottom of the left edge of the diagram. The other two traces indicate the strain on opposite sides of the middle of the column. The elongated light spots at the top of the figure are the time marks spaced 1/120 sec. apart.

At the left edge of the figure all the three indicated quantities are zero. The cam hits and the follower begins to depress the column when the three traces start to deviate from the horizontal straight line. The load increases almost linearly until it reaches a maximum; soon after it drops suddenly as the cam passes over the column. The two strains first increase uniformly as the column is compressed. After about 0·0052 sec. one of them begins to increase more rapidly while the other indicates a strain reversal. At that moment the column starts to buckle; the maximum load is reached in about 0·0062 sec.

The aluminium alloy column tested had a solid rectangular cross section of $\frac{1}{2}$ in. by $\frac{1}{4}$ in. and was 11 in. long between knife edges. Its slenderness ratio was 153, its e value about 0·001, and the Ω value of the test approximately 10. The Euler load was 558 lb. and the maximum load recorded in the test about 1,350 lb. Thus the overload factor was about 2·4.

3.6. EARLIER THEORIES OF THE RAPID LOADING OF COLUMNS

The effect of rapid load application to columns has been analysed by a number of investigators. The earliest paper on this subject known to the author is by Carel Koning and Joseph Taub[37]. In this paper the load is assumed to be applied suddenly to a pin-ended column and to continue acting on it with a constant value for an interval of time τ. At $t = \tau$ the load is suddenly removed. The axis of the column in the initial, natural, unstrained state of the column deviates slightly from

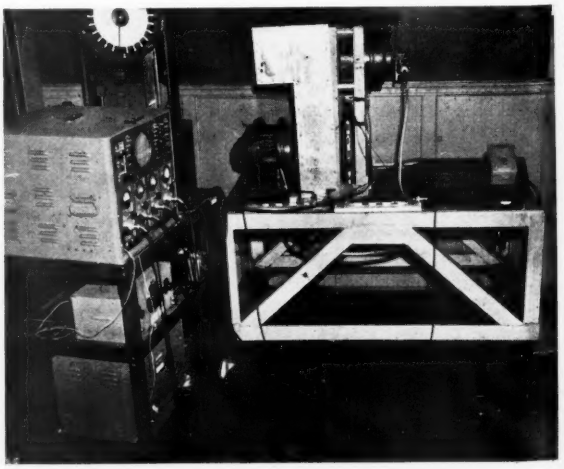


FIGURE 37. Rapid loading machine.

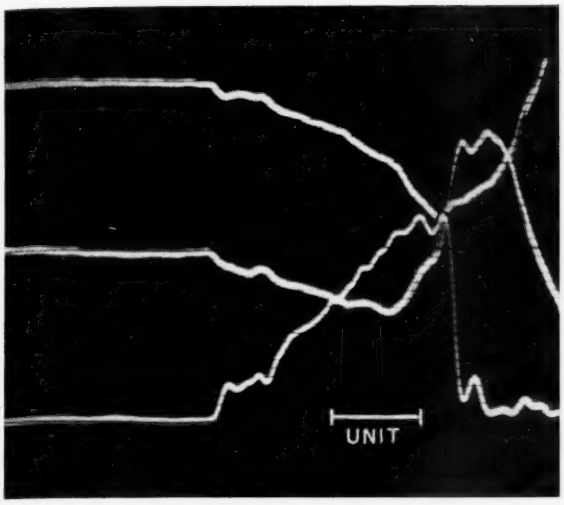


FIGURE 38. Oscillogram of rapid loading test. $L/\rho = 153$, $\epsilon \cong 0{\cdot}001$, $\Omega \cong 10$. The unit length shown corresponds approximately to 0·0025 sec., 950×10^{-6} strain, and 558 lb. load. The initial values of the strains are equal. They are projected apart for the sake of clarity.

the straight line; the deviations are assumed to follow a sine law and to comprise one or more half-wave lengths. The deflections of the column during the load application and the subsequent free vibrations are calculated and the maximum value of the curvature is determined. This maximum dynamic curvature, which is proportional to the maximum bending moment, is compared with the maximum curvature that would be caused by the same axial compressive load under conditions of static equilibrium.

It is found that the maximum dynamic bending moment is smaller than the maximum static bending moment when τ is small; and it is greater than the maximum static bending moment when τ is sufficiently large. As long as the load applied is less than the Euler load, the ratio of the maxima of the dynamic and static moments approaches 2 as τ increases indefinitely. When the load exceeds P_E, the ratio increases without bounds as τ increases.

The paper establishes firmly the important fact that loads exceeding the Euler load can be carried by a perfectly elastic column when the duration τ of the load is small. Moreover the conditions of collapse, or of large permanent deformations, of a column made of actual materials are determined approximately. The elastic stresses can be computed from the maximum value of the bending moment; when these calculated stresses reach the yield point, the column can be assumed to have failed.

The numerical results of the calculations are presented in diagrams. The principle of superposition is used to evaluate the effect of the simultaneous presence of several natural modes in the deflected shape; it is found that only the first mode is of importance when the applied load is near the Euler load. The principle of superposition is valid in this analysis because the problem is governed by linear equations.

Similar calculations were made in 1945 by J. H. Meier[38], who obviously was not aware of the work done by Koning and Taub.

Another interesting paper was published by S. Lubkin and J. J. Stoker in 1943[39]. Its subject is the investigation of the stability of a pin-ended column upon which a constant force P and a periodic force of amplitude H and frequency f^* are acting simultaneously. Lubkin and Stoker designate a combination of the quantities P, H and f^* as stable if the amplitude of the oscillations of the column remains bounded at all times. The combination is unstable when this is not the case.

The stability of the system is decided from a solution of the Mathieu equation and diagrams are presented in which the stable and unstable combinations are indicated. If the requirement for stability is that the deflections remain bounded at all times, even in the presence of small perturbations in the quantities P, H, and f^*, no column is stable in the absence of all damping. Fortunately, however, a small amount of viscous damping suffices to alter this situation. It is possible to have stability even when the steady part of the load exceeds the Euler load, but in such a case the amplitude of the periodic part must be large enough to ensure that the combination of the two be less than the Euler load, at least part of the time, and the frequency must be chosen suitably. On the other hand, a column can be unstable when P is smaller than P_E and even when it is a tensile force, provided that H and f^* have the proper values. The lack of stability in such a case represents the building up of transverse vibrations in consequence of a periodic axial force. This phenomenon is related to resonance in the presence of periodic transverse forces.

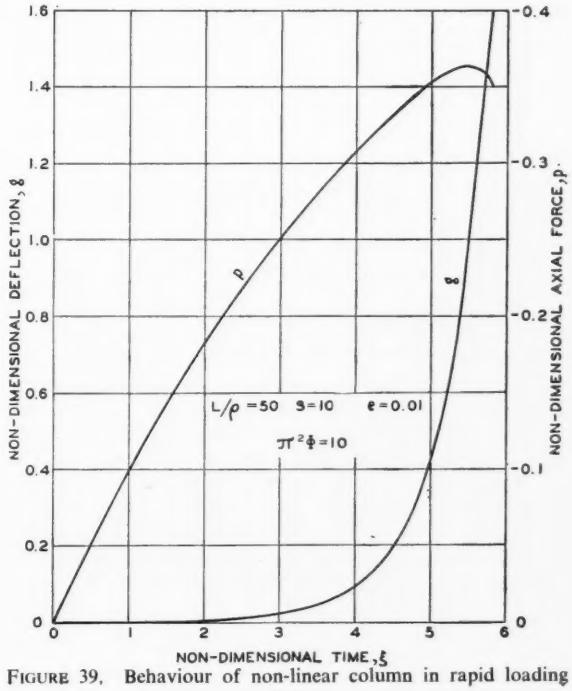


FIGURE 39. Behaviour of non-linear column in rapid loading test.

3.7. RAPID LOADING OF NON-LINEAR COLUMNS

The phenomenon explored by Koning and Taub and by Meier were governed by linear equations with constant coefficients and the problem discussed by Lubkin and Stoker led to a linear differential equation with variable coefficients. The linearity of the formulation was a consequence of the linearity of the stress-strain relationship and of the boundary conditions which stipulated the force acting upon the column. When, as in Appendix II, the end displacement of the column is prescribed rather than the force, the problem becomes non-linear even though the material is linearly elastic. This is the situation in the ordinary column test.

When the material follows a non-linear stress-strain law, this non-linearity is superimposed upon the non-linearity of the boundary conditions of the column test in the ordinary testing machine. In Appendix IV calculations are described which were done to obtain the deflection and load histories of a short column in a rapid test with the end displacement prescribed as proportional to time. The stress-strain relationship was stipulated as the cubic law discussed in Appendix III,

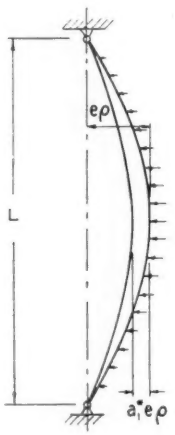


FIGURE 40. Arch deflected under transverse load.

and thus the equations of motion, equilibrium, and continuity derived in the Appendix (equations (209), (214) and (220)) were valid.

Because of the essentially non-linear character of the load-deflection relationship the equations could not be linearised, even for the small displacements occurring at the beginning of the rapid loading process. The solution was obtained therefore by expanding the deflection, the axial force, and the bending moment in infinite power series in terms of time. Fortunately the coefficients could be calculated from one cubic and from sets of two simultaneous linear equations. In the numerical work it was found convenient to take the increments of time small and to re-expand the solution successively several times. In this manner the number of terms in the series could be kept small without incurring inaccuracies of undue magnitude.

The results of a fully worked out example are shown in Fig. 39. The slenderness ratio of the column was assumed as 50, and the elasticity of the testing machine was characterised by $S=10$. This value does not suffice to produce triple-valued regions for the force as a function of the end displacement in the static solution when the stress-strain law is a cubic.

Figure 39 should be compared with Figs. 15 and 18. From these the maximum absolute value of the quantity p which is proportional to the axial load is about $-0{\cdot}215$ and the corresponding value of the non-dimensional loading head displacement ξ, is about 0·37 in a rigid testing machine. (The negative sign of p indicates compression.) In an elastic testing machine for which $S=10$ the total loading head displacement is 0·37 plus $0{\cdot}215\times 10$, that is, altogether 2·52. In the rapid loading of the non-linear column (Fig. 39) the lateral deflections are retarded by the effect of the inertia and thus at $\xi=2.52$, the compressive force in the column is still rising. The maximum is reached at $\xi=5{\cdot}4$; it is about $(0{\cdot}364/0{\cdot}215)\simeq 1{\cdot}7$ times the static value.

A comparison of Figs. 35 and 39 shows a general similarity in the displacement histories of the linear and the non-linear columns. The lateral motion of the column begins slowly and accelerates rapidly in the later stages of the loading process. The load-displacement relationship in Fig. 39 is non-linear from the outset. This is a consequence of the non-linearity of the stress-strain law.

4. The Buckling of a Transversely Loaded Flat Arch

Historically, the first example of the non-linear phenomenon of the jump of an elastic system governed by Hooke's linear law to be discussed in the engineering literature, was the buckling of a transversely loaded flat arch whose ends were restrained from displacement but were free to rotate. Solutions were obtained by Biezeno[40], Timoshenko[41] and Marguerre[42] between 1929 and 1938. More recently the problem has been studied at the Polytechnic Institute of Brooklyn by the author and by Victor G. Bruce, who investigated the dynamics of the motion of the jump phenomenon[43,44,45]. Only a few results of this investigation will be presented here inasmuch as they complement the study of the buckling process of columns, which forms the main part of this report. Another recent investigation of the problem of the flat arch is due to Y. C. Fung[46].

Figure 40 shows the flat arch whose initial shape is that of a half-sine wave. The initial amplitude of the sine wave is $e\rho$ where ρ is the radius of gyration of the cross section. The end points of the arch are attached by pin joints to points fixed in space. Under the transverse load the deviations from straightness first decrease and, when the load is large enough, start to increase in the opposite direction. The deflection of the midpoint of the arch from the initial position is denoted $a^*{}_1 e\rho$. The transverse load is distributed according to a sine law with zero intensity at the pin joints and a maximum intensity at the middle of the column. The load is characterised by the non-dimensional quantity q defined as

$$q=(1/2\pi)(L/e\rho)(Q/P_E) \quad . \quad . \quad (41)$$

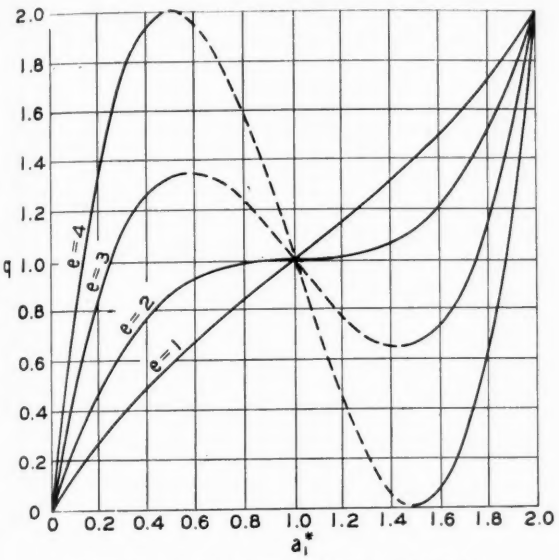


FIGURE 41. Equilibrium positions of flat arch.

where Q is the total transverse load and P_E is the Euler load of a straight column whose length, cross section, and material are the same as those of the arch.

In Fig. 41 the positions of equilibrium of the arch are presented. When the arch is comparatively flat, $e<2$, the load-deflection relationship does not differ essentially from a linear one, although the arch is weaker in the nearly straight position, that is less capable of resisting a transverse force, than in the initial position. When $e>2$, equilibrium with a given load is possible at three different displacements in a certain range of the load. Of these equilibrium positions the ones corresponding to the smallest and to the largest displacements are stable, and the intermediate position is unstable. The unstable branch of the deflection curve is shown in dotted lines.

When the load is in the range where the deflection is triple-valued, the arch may jump from the small-deflection equilibrium into the large - deflection equilibrium position. The jump is unavoidable if the load is increased from zero to a value at which no small-deflection equilibrium displacement is possible.

A large-deflection equilibrium position exists even in the absence of an external load when e is greater than 4. An example of this is shown in Fig. 42 where $e=8$. This large-deflection position of equilibrium can be reached by first increasing the load until the jump takes place and then taking the load off. Another interesting feature of the diagram is the dash-dotted ellipse. It gives the amplitude $a^*_2 e\rho$ of the second mode of deflections (two half-sine waves) as a function of the load. It follows from the theory that these amplitudes of the second mode are associated with the amplitudes of the first mode represented by the dash-dotted straight line. There exist therefore unstable equilibrium positions in which the first and second modes of deflections are mixed, and these positions correspond to smaller loads than the positions associated with the first mode alone.

The arch represented by Fig. 42 is likely to buckle in a complex fashion. First, the half-sine wave deflections increase as the load is increased. When the value of the transverse load corresponding to the intersection of the solid and the dash-dotted lines representing a^*_1 is reached, the unstable equilibrium associated with the combined mode becomes possible without a change in force or deflection. Starting from this bifurcation point, the arch follows the dash-dotted straight line rather than the full curve, unless it is prevented from doing so by some external constraint. But the equilibrium in the combined mode is possible with increasing deflections only if the force is decreased. When this is not done, the unbalanced part of the external force accelerates the arch, with the effect that the change in equilibrium takes place as a snap action.

In both the initial and the final states the deflected shape is a single half-sine wave. Nevertheless, during the jump the second mode as well as the first is present. In fact, without the second mode the jump can take place only at a higher load. Similar situations are likely to exist in the buckling process of thin shells. The observations made in connection with Fig. 40 point out the possibility that erroneous conclusions can be drawn from calculations by the energy method if the assumptions made for the deflected shape are based on the final deflections observed in experiment.

5. Creep Buckling

5.1. THE PHENOMENON OF CREEP

When structural metals are heated to high temperatures and then loaded, the phenomenon of creep can be observed. There is a sizable literature on creep; as examples, Nádai's[47] and Smith's[48] books and

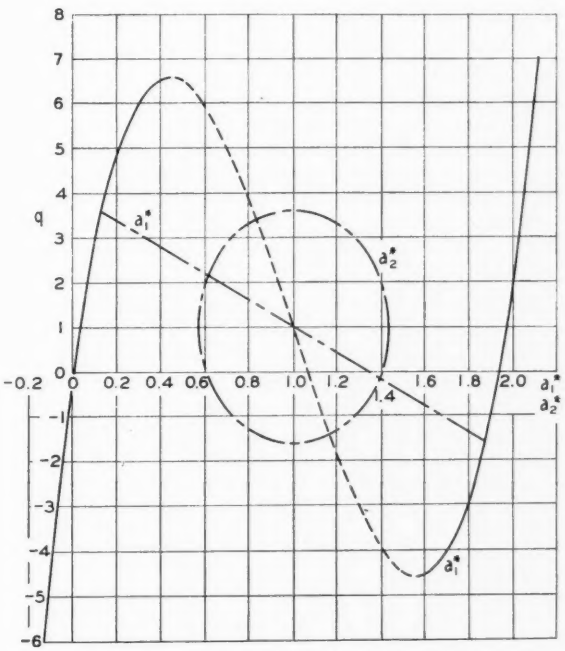


FIGURE 42. Two modes of deflection of arch.

Orowan's recent article[49] may be mentioned. In a dead weight tensile creep test, as a rule, the application of the load is followed by an instantaneous deformation which may be partly elastic and partly plastic. Subsequently the deformations continue, but at a gradually diminishing rate in a process which is often called primary creep. After a while the rate at which the deformations proceed becomes constant; this secondary or steady phase of creep is usually the most important one in engineering applications. It is followed by a third phase in which the diminishing cross section area causes the average tensile stress to increase; the ensuing increased creep rate accelerates the process until the test specimen fractures. (For an analysis of tensile creep fracture *see*, for example, the author's paper[50]).

The results of tensile creep tests are generally presented to the engineering profession in the form of charts in which time to fracture and steady creep rate are plotted against the average stress prevailing immediately after the load is applied. In a double logarithmic plot the creep rate curves are usually straight lines and can be represented by the relationship

$$\dot{\varepsilon} = (\sigma/\lambda)^n \quad . \quad . \quad . \quad (42)$$

where $\dot{\varepsilon}$ is the time rate of strain, σ is the stress, and λ and n are constants.

5.2. COLUMNS SUBJECT TO CREEP

Naturally, the creep phenomenon has an influence upon the behaviour of columns. This effect has been investigated both experimentally and theoretically. In a column test made at a high temperature the initial deviations of the centre-line of the column from the straight line can be observed to increase in consequence of the creep deformations. At the outset the displacements take place very slowly; as time goes on the rate of deflections increases gradually and finally the column fails with a snap. From the mathematical standpoint the snap process is a consequence of the non-linearity of the stress-strain rate law. When $n=1$ in equation (42), the deflections of the column increase gradually and approach infinity only as time approaches infinity; this was shown by the author and by Joseph Kempner in an earlier paper[51]. As far as is known to the author, creep buckling was first treated theoretically by A. M. Freudenthal[52]; his results were further discussed by Kempner and Pohle[53].

It can certainly be said when the column snaps through and fails, that it buckled. But this buckling phenomenon can by no means be analysed with the aid of Euler's criterion of stability. In fact, no characteristic load can be assigned to this type of instability because the column buckles under any (non-zero) compressive load if the load is applied for a long enough time. The quantity of interest in creep buckling is the critical time at which failure occurs; this depends, of course, both on the magnitude of the load and the initial deviations of the axis of the column from straightness. The essentials of column behaviour in the presence of linear creep were discussed by the author on the basis of a simple model of a column in an earlier publication[54]; here an analysis will be presented of a column

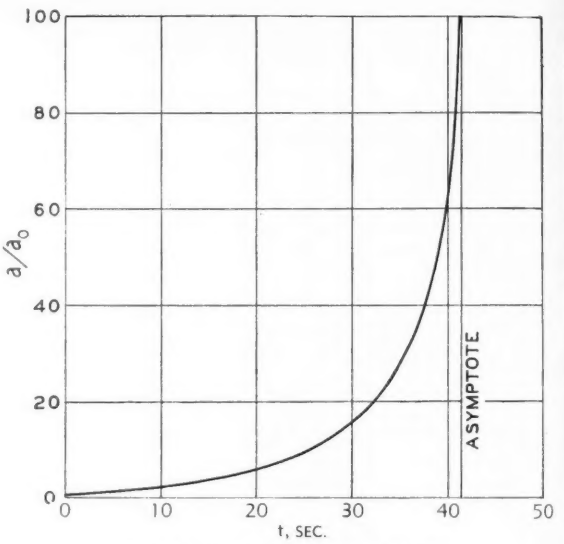


FIGURE 43. Creep buckling of column.
$\dot{\varepsilon} = (\sigma/10000)^3$/hr $L/\rho = 100$ $e = 0{\cdot}01$ $E = 6{\cdot}5 \times 10^6$ lb./in.^{2}
$P/P_E = 0{\cdot}75$

of idealised I-section when the deformations are governed by the creep law of equation (42).

First, it should be mentioned that more complete laws of deformation can be used in the analysis to take into account the elastic strain as well as the creep strain. This was done by Kempner and others[55]. Among other investigators of creep buckling may be mentioned A. D. Ross[56], Joseph Marin[57], L. R. Jackson, A. D. Schwope and F. R. Shober[58], Charles Libove[59], George Gerard[60], F. R. Shanley[61], T. P. Higgins, Jr.[62], D. Rosenthal and H. W. Baer[63] and once more Charles Libove[64].

In a recent paper[65] the author showed that the contribution of the elasticity of the material to the state of stress and strain in the body after a moderately large amount of creep took place is usually insignificant. Thus good approximate results can be obtained in many cases if the elasticity of the material is disregarded and if the calculations are made on the assumption that the deformations are completely governed by the creep law of equation (42).

5.3. ANALYSIS OF CREEP BUCKLING

Because of the similarity between the creep law of equation (42) and the stress-strain law of equations (14) and (147) adopted in the study of the behaviour of the non-linear perfectly elastic column, a considerable portion of the column analysis presented in earlier parts of this paper and in Appendix III can be used in the calculation of creep buckling. Thus differentiation of both members of equation (167) with respect to time yields for the rate of change of the radius of curvature of the column

$$(\partial/\partial t)(1/r) = -(h^*/2L^2)(\partial^2 \dot{w}/\partial x^2) \quad . \quad (43)$$

where $\dot{w}$ is the time rate of change of the non-dimensional lateral deflection of the column. Similarly,

equation (158) becomes after differentiation with respect to time

$$(\partial/\partial t)(1/r)=(\dot{\varepsilon}_t-\dot{\varepsilon}_c)/h^* \quad . \quad . \quad (44)$$

where $\dot{\varepsilon}_t$ and $\dot{\varepsilon}_c$ are the rates of change of the strain in the flanges of the column on the convex and the concave sides, respectively. It follows then that

$$\dot{\varepsilon}_t-\dot{\varepsilon}_c=-[2/(L/\rho)^2](\partial^2\dot{w}/\partial x^2). \quad . \quad (45)$$

If the non-dimensional force p and moment m are defined as

$$p=2P/A\lambda$$
$$m=4M/A\lambda h^* \quad . \quad . \quad . \quad (46)$$

replacement of equation (159) by

$$\sigma=\lambda\,\dot{\varepsilon}^{1/n} \quad . \quad . \quad . \quad (47)$$

and manipulations similar to those performed between equations (159) and (164) result in

$$\dot{\varepsilon}_t=(1/2)^n(p+m)^n$$
$$\dot{\varepsilon}_c=(1/2)^n(p-m)^n. \quad . \quad . \quad . \quad (48)$$

If the deformations take place so slowly that the inertia forces can be neglected, the condition of static equilibrium expressed in equations (169) and (170) holds unchanged. Thus

$$\dot{\varepsilon}_t=(1/2)^n p^n(1-w)^n$$
$$\dot{\varepsilon}_c=(1/2)^n p^n(1+w)^n. \quad . \quad . \quad (49)$$

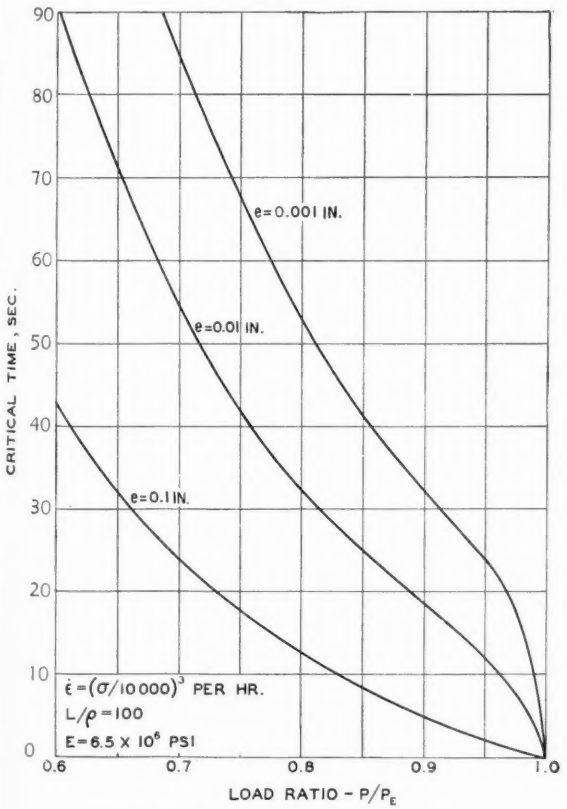


FIGURE 44. Critical time in creep buckling.

Substitution in equation (45) and manipulations yield

$$(\partial^2\dot{w}/\partial x^2)+\Pi[(1-w)^n-(1+w)^n]=0 \quad . \quad (50)$$

with $$\Pi=(1/2)(L/\rho)^2(p/2)^n. \quad . \quad . \quad (51)$$

This differential equation will now be solved for the case that $n=3$. Then

$$(\partial^2\dot{w}/\partial x^2)=2\Pi(3w+w^3). \quad . \quad . \quad (52)$$

On the basis of the experience gained in the analysis of the non-linear perfectly elastic column, only the first term of the sine series

$$w=a\sin\pi x+a_2\sin 2\pi x+a_3\sin 3\pi x+\ldots \quad (53)$$

where the coefficients a_n are functions of time, will be retained. In such a manner one arrives at the differential equation

$$(da/dt)=-(3\Pi/2\pi^2)(4a+a^3). \quad . \quad (54)$$

This equation can be solved for t through separation of the variables:

$$t=-(2\pi^2/3\Pi)\int_{a_0}^{a} da/[a(4+a^2)]. \quad . \quad (55)$$

As it is known that

$$\int da/[a(4+a^2)]=(1/8)\log[a^2/(4+a^2)]+C \quad (56)$$

the time t necessary to reach any arbitrary displacement a can be computed.

Figure 43 shows the development of the deflections with time for a column whose modulus of elasticity is $6{\cdot}5\times10^6$ lb./in.^{2} and the deformations of which are governed by the creep law $\dot{\varepsilon}=10^{-12}\sigma^3$. The slenderness ratio is assumed as 100 and the non-dimensional initial deviation amplitude $e=10^{-2}$. The figure represents the creep buckling process under a compressive load which is 75 per cent. of the Euler load. The conditions chosen are similar to those prevailing with an aluminium alloy column at 600°F. It can be seen from the figure that the column snaps through in about 41 sec. after the load application, even though the load is considerably less than the Euler load.

Of particular interest is, of course, the critical time t_{cr} defined as the time necessary for the column to reach indefinitely large deflections. If a_0 denotes the non-dimensional initial amplitude of the total deflections (at $t=0$), integration from a_0 to infinity gives

$$t_{cr}=-(\pi^2/12\Pi)\log[(4+a_0^2)/a_0^2]. \quad . \quad (57)$$

If the instantaneous deformations developing at the time of the load application are considered perfectly elastic, the amplitude a_0 can be calculated from the amplitude e of the deviations from straightness of the unloaded column (assumed to be of half-sine-wave shape) with the aid of the relation

$$a_0=e/[1+(P/P_E)] \quad . \quad . \quad (58)$$

where P_E is the Euler load of the linear elastic column

$$P_E=\pi^2EI/L^2 \quad . \quad . \quad . \quad (59)$$

and P is counted positive when it is a tensile force. In practical applications a_0 should always be small as compared to unity; hence

$$(4+a_0^2)/a_0^2 \simeq (2/e)^2\,[1+(P/P_E)]^2 \quad . \quad (60)$$

Further simple substitutions yield

$$t_{cr} = -(2\pi^2/6)(\rho/L)^2[\lambda/(P/A)]^3 \log\,\{(2/e)\,[1+(P/P_E)]\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

This can also be written in the alternative form

$$t_{cr} = -(\varepsilon_{cr}/\dot{\varepsilon}_{nom}) \log\,\{(2/e)\,[1+(\sigma_{ave}/\sigma_E)]\} \quad (62)$$

where $\varepsilon_{cr}=(\pi^2/3)/(L/\rho)^2$ is the critical strain of a non-linear perfectly elastic column following the cubic stress-strain law, $\sigma_{ave}=(P/A)$ is the average stress in the column, counted positive when tensile, $\sigma_E=\pi^2E/(L/\rho)^2$ is the Euler stress of the column corresponding to its state at the moment of load application, E is the modulus of elastic deformations at the moment of load application, and $\dot{\varepsilon}_{nom}=(\sigma_{ave}/\lambda)^3$ is the nominal rate of creep strain corresponding to the average stress σ_{ave}.

We may also write

$$t_{cr} = -(\varepsilon_{cr}/\dot{\varepsilon}_{nom}) \log\,(2/a_0) \quad . \quad . \quad (63)$$

where a_0 is the non-dimensional amplitude of the deviations from straightness, assumed to be in the shape of a half sine wave, of the axis of the column at the moment when the creep deformations begin.

As P and $\dot{\varepsilon}_{nom}$ are negative when the column is compressed, all these formulas yield positive values for the critical time.

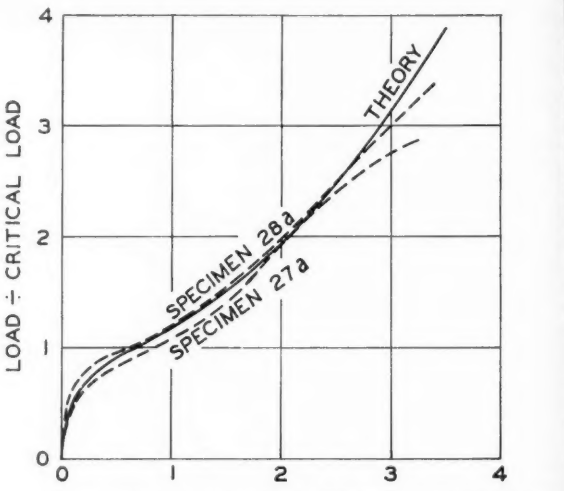
5.4. DISCUSSION OF RESULTS

Two important conclusions can be drawn from the formulas derived for the critical time (*see* Fig. 44). First, the time is comparatively little influenced by the initial crookedness of the column; an increase in the initial amplitude e from 0·001 to 0·01 results in a decrease in the lifetime of the column by a factor of 0·62 only when $P/P_E=-0{\cdot}8$. A further increase to 0·1 results in a further factor of 0·385. This is fortunate because uncontrollable inaccuracies of manufacture do not affect unduly the usefulness of the column. Second, t_{cr} is very sensitive to the magnitude of the applied average compressive stress. If the load is decreased from 90 per cent. of P_E to 75 per cent., the lifetime of the column is more than doubled for all the e values shown in the figure. The consequence of this fact is that the designer can greatly increase the safety of his structure by increasing only slightly the cross-sectional area of the column.

Obviously equations (61), (62) and (63) must not be used when a_0^2 is of the order of magnitude of 4. However, in such a case equation (57) is still valid; it can also be written as

$$t_{cr} = -(1/2)(\varepsilon_{cr}/\dot{\varepsilon}_{nom}) \log\,[(4+a_0^2)/a_0^2]. \quad . \quad (64)$$

The right-hand member of this equation vanishes when a_0^2 approaches infinity; this is the case when the compressive end load $-P$ approaches the Euler load P_E, as can be seen from equation (58).



(*Courtesy of A.S.M.E.*)

FIGURE 45. Behaviour of rectangular initially slightly curved panel with four edges simply supported when subjected to edgewise compression.

It is also of interest to discuss the behaviour of the column when its material follows a linear creep law. With $n=1$ equation (50) becomes

$$(\partial^2\dot{w}/\partial x^2) = 2\Pi w. \quad . \quad . \quad . \quad (65)$$

In this case the initial sinusoidal shape is maintained rigorously during the creep deformations. Consequently the assumption

$$w = a \sin \pi x \quad . \quad . \quad . \quad (66)$$

results in the equation

$$\dot{a} = (da/dt) = -(2\Pi/\pi^2)\,a. \quad . \quad . \quad (67)$$

Separation of the variables and integration yield

$$t_1 - t_0 = -(\pi^2/2\Pi) \log\,(a_1/a_0). \quad . \quad (68)$$

This expression represents the time necessary for the column to deflect from a_0 to a_1; an indefinitely large deflection a_1 can be reached therefore only after the lapse of an indefinitely large time, provided a_0 is finite. Thus the concept of the critical time is meaningless with a linear viscous column.

Finally, it is noted that the derivations given in this Section are correct only if n is an odd integer. When this is not the case, the stress-strain relationship for tension must be prescribed separately from that valid for compression.

6. What is the Buckling Load?

6.1. EULER'S THEORY AND THE ACTUAL BEHAVIOUR OF SLENDER COLUMNS

If all columns and other slender structural elements behaved in the simple manner predicted by Euler, the answer to the question "What is the buckling load?" would be a trivial one. The column, thin plate, and the thin-walled cylindrical shell would shorten under the applied compressive load, but otherwise they would not change their shape below the critical value of the load; and when that value is reached, they would sud-

denly develop large deformations of a type not observed earlier. Under such conditions the critical load of the classical theory would obviously be accepted by both theoreticians and experimental men as the buckling load.

The analyses and the experiments described in this paper have shown that the actual buckling process is much more complex and varied than the one predicted by the classical theory of perfect columns. The agreement between classical theory and reality is best in the case of the slow loading of a very slender column in the conventional testing machine. Such a column does not remain straight as the load increases because the unavoidable deviations of its axis from the straight line begin to increase as soon as a compressive axial load is applied to it. But the magnification of the deviations is hardly noticeable at the outset, provided that the column is carefully machined and centred, and becomes very large when the Euler load is approached within about one per cent. In a column test the load increases almost in direct proportion to time until the maximum load is reached. Afterwards the load remains constant for a considerable time while the column buckles visibly.

6.2. SHORT COLUMN AND FLAT PLATE IN THE ORDINARY TESTING MACHINE

The situation is not so clear-cut when a short column is compressed in a very rigid testing machine. The increases in load lag more and more behind those that correspond to proportionality, a maximum value is reached, and the load begins to decrease as the loading head of the testing machine descends farther. The only value of the load with a distinguishing property is the maximum of the load; no straight experimentalist would ever hesitate to put down this maximum in his test log as the buckling load. Yet scientifically-trained aeronautical engineers are likely to object to such a definition. They know that the buckling load and the maximum load are not the same in edgewise compression tests of thin rectangular panels.

This was first observed experimentally by Schuman and Back at the National Bureau of Standards in 1930[66] and was explained theoretically by Th. von Kármán in 1931[67]. Thin panels develop bulges so gradually that their beginning cannot be observed. The bulges are always visible by the time the critical load is reached, but they continue to grow as the loading head of the testing machine descends farther. At the same time the total load acting on the panel also keeps increasing because the portions of the panel adjacent to the supported edges are restrained from bulging and are capable of developing higher stresses. The connection between compressive load and maximum lateral deflection is illustrated in Fig. 45. The theoretical curve shown was calculated by J. M. Coan[68] for an initial deviation amplitude of one-tenth of the thickness of the panel, and the experimental curves were obtained with cross-laminated Fibreglas panels at the Polytechnic Institute of Brooklyn[69].

From the standpoint of the experimental man there is little choice left as to the buckling load when a well machined and centred short column is tested in an elastic testing machine. He cannot fail to call that load the buckling load under which the column buckles in an explosive manner. The theory derived in Appendix III indicates that at the latest the column must snap through shortly after the maximum load is observed in the column test. It is likely to snap just before the maximum load of the static solution is reached and it may snap any time after the loading head displacement is so large that three load values, rather than one, correspond to it in Fig. 18. The time of buckling and the exact value of the buckling load depend on the elasticity of the testing machine and on the magnitude of the disturbance to which the column in its quasi-static state of equilibrium is subjected. The diagrams drawn in the phase plane (Figs. 21 to 23) show indubitably that the equilibrium of a column may be perfectly stable for small disturbances and yet unstable when the disturbances are sufficiently large.

The numerical values of the disturbance that are necessary to cause the column to snap through are naturally correct only for the cubic stress-strain law upon which the calculations are based. They indicate that in a smoothly run test the snap-through process cannot very well take place much before the maximum of the static curve is reached. A similar conclusion can be drawn from Frank J. Romano's study[27] of the model of a column whose material has the properties of 24S-T aluminium alloy, although the necessary disturbance there is somewhat smaller than the one derived in this paper. Fig. 15 shows that inaccurate columns fail under loads considerably smaller than accurate ones.

6.3. THE BUCKLING OF THIN SHELLS

Although a complete dynamic analysis of the buckling of a thin-walled circular cylindrical shell subjected to axial compression is still lacking, a great deal of interesting information can be had from the solution of the static problem. As is well known, the classical approach yields $0{\cdot}6E\,(t^*/r^*)$ for the critical stress when the buckled shape is axially symmetric, as well as when it is multilobed around the circumference. This was derived by Lorenz, Timoshenko and Southwell between 1908 and 1914[70, 71, 72]. However, experimental results scatter a great deal (*see* References 73 to 75) and agree better with the theory if the constant 0·6 is replaced by 0·3. In 1932 an explanation for the discrepancy was proposed by W. Flügge[76], who investigated the effect of deviations from the perfect cylindrical shape. To verify the observation made in the experiments that cylinders with high values of the r^*/t^* ratio require smaller values of the constant (sometimes below 0·2) than cylinders having low r^*/t^* ratios, Donnell advanced a theory in his paper[75] in which the initial deviations from the perfect shape were discussed with the aid of a large-deflection theory.

A deeper insight into the behaviour of thin shells was obtained between 1939 and 1942 when Th. von Kármán and H. S. Tsien published a series of papers[77, 78, 79, 80] dealing with the large-deflection equilibrium configuration of shells. They found that the essential difference between the flat and the curved panel was the non-linear load-deflection relationship in

the latter. In consequence equilibrium was possible with both small and large deflections for the same overall shortening of the cylinder when the shortening was large enough. The small deflections were associated with a high, and the large deflections with a low, value of the compressive load. It was conjectured that a change in the equilibrium configuration prompted by some disturbance could take place anywhere in the range of overall deflections where the load was multi-valued. H. S. Tsien[80] also proposed tentatively that the change was likely to occur in dead-weight loading when the total potential of the system in the two configurations was the same, and in rigid testing machine loading when the strain energy was the same.

Further advances in this theory were made by D. M. A. Leggett[81], Herman F. Michielsen[82] and Joseph Kempner[83]. Fig. 46 is redrawn here from Kempner's report. The main difference between this curve and the ones corresponding to $S=100$ in Fig. 18 is that the curve for the shell is triple-valued in the absence of any elasticity in the testing machine, while the curves for the column in the rigid testing machine ($S=0$) are single-valued functions $p(\xi)$. It can be concluded therefore that the axially compressed cylinder must snap through even if the testing machine is rigid.

The empirical fact that the maximum value of the static curve is never reached with the shell, indicates that some theoretical reason must exist for the occurrence of the snap-through process at lower loads. As the small-deflection stable branch and the unstable branch of the curve are almost parallel and very close together, the elasticity of the testing machine is likely to reduce materially the magnitude of the disturbance necessary to trigger the snap. Thus it is probable that both the elasticity of the testing machine and the initial deviations from the perfect shape have a greater influence with the shell than they do with the non-linear column. This would explain the large scatter observed with the shell.

Another detailed and independent investigation of the stability of thin shells is contained in W. T. Koiter's doctoral thesis[84]. The von Kármán-Tsien theory of the buckling of spherical shells was further discussed by K. O. Friedrichs[85]. Recently H. L. Donnell has extended his earlier work on the effect of initial imperfections on the buckling of cylindrical shells[86].

6.4. THE CUSTOMARY DESIGNATION OF THE BUCKLING LOAD

If the significant features of the preceding discussion are summed up, it may be said that the buckling load observed with a slender column when it is loaded slowly in the ordinary testing machine is, for all practical purposes, the critical load of the classical theory of stability. When the loading takes place very rapidly, this is obviously not true because the load reaches values far exceeding the critical load. A thin, rectangular, perfectly elastic plate supported along all four edges develops bulges so innocuously that the buckling load loses its physical significance; with such a plate the only load that counts is the collapse load, which is considerably higher than the critical load. Finally, the

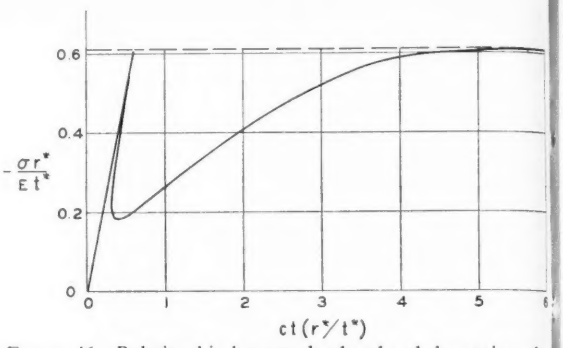


FIGURE 46. Relationship between load and end shortening when thin cylindrical shell is compressed axially.

axially compressed thin shell collapses at about one-third of the critical load.

It is an oddity of accepted terminology that engineers are in the habit of considering as the critical load of the cylinder the experimentally observed buckling load, that is the lowest load at which a sufficiently large disturbance is likely to provoke the snap-through process. In this case they are therefore perfectly willing to divorce the definition of buckling from the Eulerian theory of stability. At the same time, they cling to the classical stability concept for the flat plate and refuse to accept the maximum load as the buckling load.

6.5. THE DUBERG-WILDER DEFINITION

The lack of consistency in terminology and accuracy in the definition becomes even more apparent when the material is not perfectly elastic. This has been the cause of the controversy over the tangent modulus load as the correct stability limit. The controversy began with the publication of Shanley's two papers already mentioned[20]. In 1950 Duberg and Wilder discussed the problem in an interesting article[87] in which they defined the critical load as the load at which the column starts to deflect.

A number of objections can be raised to this definition. When the column is initially perfectly straight (as was assumed with this definition), it will never deflect laterally unless there is some transverse disturbance in its equilibrium configuration. On the other hand, a small but finite disturbance can cause permanent deflections even if the load is below the tangent modulus load; as a matter of fact a sufficiently large lateral disturbance can cause permanent deflections during the loading process or under a stationary load at any level of the load. Finally, if the column has any initial deviations from straightness, as it always has in practice, then deflections begin just as soon as the loading starts. (*See* the author's discussion of the paper in Reference 88.)

It is true that the somewhat vague engineering definition of the buckling load was refined by Duberg and Wilder when they derived the behaviour of the perfectly straight column as the limit of the behaviour of the initially slightly curved column when the initial curvature approaches zero. It is questionable whether a mathematically rigorous limiting process is possible

in this case. Nevertheless a physical interpretation of a family of curves with various initial deviations from straightness appeals to the imagination of the engineer.

Irrespective of its accuracy and consistency, this definition was obviously "tailor-made" for the short column subjected to the ordinary compression test. It cannot be used for the rapid loading of columns, for viscous columns, or for thin shells subjected to axial compression, and thus it has no claim to generality. This being the case, it would be much simpler to define the critical load directly as the tangent modulus load and to use the analysis of the equilibrium of the column as a justification for the definition. Any definition is admissible if it is consistent with the generally accepted principles of mechanics; it is particularly suitable if it has some usefulness. The definition of the critical load of the ordinary column test as the tangent modulus load obviously fulfils these two requirements.

The contribution made by Duberg and Wilder in the two papers mentioned lies indeed in the detailed analysis of the compression test on a static basis, neglecting inertia effects, and not in the definition of the critical load. Their work was continued at the N.A.C.A. by Wilder, Brooks and Mathauser, who produced in 1953 detailed charts illustrating the behaviour of idealised I-section columns in the ordinary column test[89]. The computations were made on an electronic computing machine. The material properties were defined according to the Ramberg-Osgood suggestions[90]. It was found that a column which has a very small initial deviation from straightness can always support a maximum load greater than the tangent modulus load. The maximum load of a column which initially is not perfectly straight can be greater or smaller than the tangent modulus load, depending on the initial deviations from straightness, the slenderness ratio, and the shape of the stress-strain curve.

6.6. THE PEARSON-PRAGER CRITERION

In a recent doctoral thesis written under Professor W. Prager's supervision at Brown University[91], C. E. Pearson suggested the following criterion of buckling:

"A mechanically idealized system will be said to be unstable if there exists a small distance ε such that for any $\delta > 0$ it is possible to find a force distribution of intensity less than δ which will produce an eventual displacement of the system in which at least one material point is distant ε from its original position. A stable system is one that is not unstable."

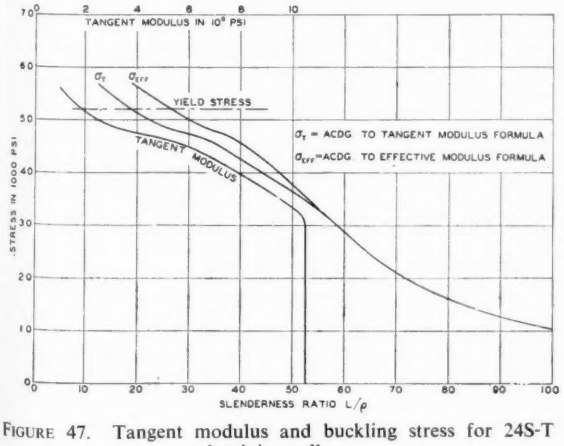

TANGENT MODULUS IN 10⁶ PSI
YIELD STRESS
TANGENT MODULUS
σT = ACDG TO TANGENT MODULUS FORMULA
σEFF=ACDG TO EFFECTIVE MODULUS FORMULA
STRESS IN 1000 PSI
SLENDERNESS RATIO L/ρ

FIGURE 47. Tangent modulus and buckling stress for 24S-T aluminium alloy.

In a much less precise manner the engineer would express the same thought by saying that a system is unstable if the application to it of a very small force can produce a very large displacement. The emergence of such an entirely different criterion illustrates the point that there is nothing sacred about the classical definition of stability. The engineer can discard it if he can replace it by a new definition that better suits his needs. Whether the Pearson-Prager criterion will be generally adopted by engineers, only the future can tell. It agrees with the classical criterion when it is applied to the perfectly elastic column.

The Pearson paper is part of the literature on inelastic buckling initiated by Shanley[20]. Other investigators who contributed to it are C. T. Wang[92], T. H. Lin[93], and P. Cicala[94].

6.7. EXPERIMENTAL PROOF OF THE CORRECTNESS OF THE TANGENT MODULUS LOAD

It has often been stated that the tangent modulus load is the correct buckling load, rather than the reduced modulus load, because column tests agree better with the former than with the latter. This reasoning is not admissible on several counts. First, the experimental buckling load of a column has not been defined in a manner acceptable to all research men. In the absence of evidence to the contrary it must be assumed that the available experimental data are maximum load values. But the maximum load is not the same as the critical load of the classical theory of stability; this follows from the investigations of Shanley[20], Duberg and Wilder[87], and the present author. The column test in the ordinary testing machine cannot be considered therefore as the touchstone of the correctness of a theory of the buckling of short columns.

A further objection can be raised on the grounds that most of the data available from column tests are not accurate enough. They were generally obtained from tests made with flat ends. The flat end condition corresponds, according to the experimental evidence, to an end fixity coefficient value ranging from 3·2 to 4·05 for slender columns. Although this variation is smaller for short columns, it introduces an uncertainty in the interpretation of the results. The error arising from this uncertainty is of the same order of magnitude as the differences between the tangent modulus and the reduced modulus loads, as may be seen from Fig. 47. (The material properties were taken from Reference 95.)

6.8. FREEDOM IN THE SELECTION OF A DEFINITION

What, then, is the buckling load of a short column in the ordinary column test? In the author's opinion no answer can be given to this question on the basis of the classical concepts of stability. The classical definition deals only with systems upon which prescribed loads are acting. The buckling load in the testing machine has to be defined first; otherwise it makes no sense. From the standpoint of the practical engineer,

three suitable definitions are available for this purpose: The reduced modulus load, the tangent modulus load, and the maximum load. The first has the advantage that it is the critical load of the classical theory. Its disadvantage is that the maximum load is always less than the reduced modulus load.

The tangent modulus load has the merit that it can be easily calculated as soon as the stress-strain curve is known. It also coincides, on the average, with the maximum load observed with actual, imperfect columns.

Undoubtedly, the maximum load has the greatest physical significance of the three loads mentioned. It is also the easiest to determine experimentally. A theoretical evaluation on the basis of the stress-strain curve has also become possible since the static maxima were calculated and published in Reference 89; moreover, in the present paper they were shown to agree well with the maxima in the dynamic loading process at low speeds.

It is important, however, to emphasise that the engineer is perfectly free to make his choice among these definitions. There is no single definition which he is compelled to accept on physical grounds.

6.9. CRITICAL LOAD VERSUS BUCKLING LOAD

This discussion should have exhausted the topic of defining the buckling load of a short column in the ordinary testing machine. Whichever definition is chosen, however, it will not satisfy the needs of the engineer in all kinds of loading and with every type of structural element. Rapid loading by prescribed forces or with prescribed displacements, repeated loading, loading in the presence of creep deformations, and other processes will all have to be investigated individually.

In this connection one point in nomenclature merits some attention. There is a definite difference in principle between the buckling load observed in a loading process where the loads keep changing, and the buckling load calculated from the classical theory, or from its generalisation to be discussed later, which always refers to a system with prescribed loads. As the latter is based on theory and is obtained as the critical, characteristic, or Eigen value of the load at which neighbouring, slightly deflected equilibrium configurations exist simultaneously with the initial configuration of deflections, it is properly called the critical load.

In the process of buckling in the testing machine, in the static or dynamic testing of the aeroplane, and in the failure of the structure in actual use, one is confronted with the physical aspects of buckling. The load at which a structure is seen to buckle should be designated preferably as the buckling load.

The compound term critical buckling load is unnecessary and should be avoided. It may have originated from the observation that theory predicts several buckling loads corresponding to different deflection patterns. In an experiment, however, only one buckling pattern is observed, namely the one that corresponds to the lowest buckling load. This load is not more critical than any of the higher ones; if it is desired to give it a special designation, it should be called the lowest buckling load.

6.10. THE DYNAMIC CRITERION OF STABILITY

Most of this paper has been devoted to buckling during the loading process. But the curves in Figs. 18 and 43 indicate the need also for a generalisation of the classical definition of buckling under prescribed loads. The classical concept of stability is of a dynamic origin. It is based on the observation that a small disturbance in the deflected shape of a lightly loaded elastic column causes vibrations in the immediate vicinity of the equilibrium position, while the same kind of disturbance leads to ever-increasing deflections and to eventual collapse when the load acting upon the column is large. The system is designated as a stable one in the first case and as an unstable one in the second case. Under the conditions of stable equilibrium the column eventually returns to its initial equilibrium configuration because the vibrations are damped out*.

A static criterion can be derived without difficulty from this observation. Under the limiting load between the stable and the unstable ranges a small disturbance cannot be magnified or diminished; the column must be able to remain in equilibrium in the disturbed configuration. Consequently there must exist equilibrium configurations in the vicinity of the initial configuration. This co-existence of neighbouring equilibrium positions has been utilised as the static criterion of buckling.

The classical static condition has been most productive in the development of the theory of stability and undoubtedly it will always remain the most important tool in the calculation of buckling loads. But it has its limitations. It cannot predict the buckling of the axially compressed thin-walled cylinder under loads smaller than the critical load of the classical theory, and it is incapable of dealing with columns subject to creep. In all such cases the analyst has to revert to the fundamental concept of stability and has to investigate the motion of a system following a disturbance.

This course was advocated by the author in his paper read before the Symposium on Engineering Structures organised by the Colston Research Society at the University of Bristol in September 1949[96]. The dynamic criterion of stability was applied to single-degree of freedom systems with dry friction, with an elasto-plastic restoring force, and with a visco-elastic restoring force, and the motion following a disturbance was calculated from the equations of motion. It was stated that

> " for the engineer stability means that he need not worry whether the equilibrium of the structural element or part of machinery he designed will be upset by some disturbance. He can rely upon the equilibrium. This statement is seldom equivalent to the concept of stability in the mathematical theory of elasticity.
>
> " The nature and the magnitude of the disturbance must be established from a statistical investigation of the conditions under which the structural element or the part of machinery will be used. The safety of the system can be safeguarded if it is made stable for all disturbances which have a probability greater than a required minimum. The system can be considered safe (that is stable) if it regains its original state of equilibrium sometime after the disturbance is over; alternatively it may be sufficient to require that the

*This is strictly correct only if the damping is viscous.

system return to a state adjacent to the original state of equilibrium.

" The expression ' sometime ' was inserted in the last sentence because vibrations can be expected to occur after the disturbance. They will be damped out only after the lapse of some time. As a matter of fact, in a perfectly elastic system the vibrations will continue indefinitely. This, however, is not a restriction of the validity of the criterion proposed because the real systems of engineering are never perfectly elastic. Internal friction in the material, friction in the attachments, and aerodynamic resistance are always present to provide damping. At the same time Coulomb type friction may stop the motion before the original state of equilibrium is completely regained. This necessitates the liberalisation of the requirement as was done in the second half of the last sentence of the preceding paragraph.

" Naturally the word ' adjacent ' needs to be defined more precisely, but this cannot be done in a general way. The magnitude of the permissible deviation of the final state from the original one depends, among other things, upon the number of disturbances of the kind and magnitude considered probable during the lifetime of the structure or machinery."

The perturbation equation used earlier in the present paper was essentially a dynamic criterion in the investigation of the stability of the loading process. It revealed that in the absence of friction the amplitude of the vibrations of a perfectly elastic column could not be magnified by a factor greater than 3; when friction is present, the high-frequency oscillations are likely to be damped out rapidly. On the other hand, one branch of the curve of equilibrium states in Fig. 18 was found unstable when the perturbation solution was obtained for the short column. The solution was in the form of hyperbolic functions rather than trigonometric functions, indicating that the amplitude of the oscillations could grow without bounds, at least until the validity of the linearised solution became lost.

By its nature, the perturbation solution can deal only with small disturbances. The effect of a finite disturbance of any magnitude upon the motion of the short column is given in all detail by the curves of Figs. 21 to 23. The magnitude of the disturbance necessary to cause the column to snap from its initial equilibrium position into another one can be read from these figures; they also show that all disturbances lead to small vibrations about the initial state of equilibrium if the displacement of the loading head of the testing machine is sufficiently small.

Finally it is of interest to mention that Professor Hans Ziegler of the Polytechnic Institute of Zurich has also advocated the use of a dynamic criterion of stability[97]. He found that the static criterion was insufficient and often even misleading with a perfectly elastic system when the loads were non-conservative. An example illustrating this point was given by Max Beck[31].

6.11. CONCLUSIONS

A short column compressed in the testing machine with the distance between its two ends fixed can be stable when the disturbance is small, and unstable when the disturbance is large; consequently it may buckle before its stability limit is reached in the sense of the classical small-deflection theory. A column whose material is subject to creep is unstable under a compressive load of any magnitude, but it may not buckle during the lifetime of the structure of which it is a part. Buckling under a static load following a disturbance has many aspects differing significantly from buckling during a loading process, particularly when the latter is rapid, and there are material differences between the various kinds of loading processes.

For these reasons it is desirable to distinguish between the stability of a fixed system in the classical sense and the stability of a loading process. There is no reason to believe that an arbitrarily chosen loading test will furnish values that can prove or disprove the classical theory of static stability. There is even a great deal of arbitrariness connected with the definition of the buckling load in a loading process.

It seems advisable to reserve the terms critical load and characteristic value of the load for the designation of theoretical loads at which the state of equilibrium is of a special character. Thus the load at which the limit of stability is reached in accordance with the classical small deflection theory, or even according to a large deflection theory, is properly called a critical load. Common sense requires that the load at which a column or a thin-walled cylinder buckles suddenly in a loading process should be designated as the buckling load.

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The final tracings of the diagrams were prepared by Mr. Matthew G. Forte.

To all these persons the author wishes to express his appreciation.

Appendix I

Original Text of Quotations

(*a*) *From "De Architectura, Libri Decem, Cum Commentariis Danielis Barbari, Electi Patriarchae Aquileiensis, Venetiis, M.D.LXVII," p. 123.*

"Ibiq; templa Deorum immortalium constituentes coeperunt fana aedificare et primum Apollini Panionio aedem (uti viderant in Achaia) constituerunt, et eam Doricam apellauerunt, quod in Doricon ciuitatibus primam factam eo genere uiderunt. In ea aede cum uoluissent columnas collocare, non habentes symmetrias earum, et querentes quibus rationibus efficere possent, uti et ad onus ferendum essent idoneae, et in aspectu probatam haberent uenustatem, dimensi sunt uirilis pedis uestigium, et cum inuenissent pedes sextam partem esse altitudinis in homine, ita in columnam transtulerunt, et qua crassitudine fecerunt basim scapi, tantum eam sexies cum capitulo in altitudinem extulerunt, ita Dorica columna uirilis corporis proportionem, et firmitatem, et uenustatem in aedeficijs praestare coepit.

"Item postea Dianae constituere aedem, querentes noui generis speciem, ijsdem uestigijs ad muliebrem transtulerunt gracilitatem, et fecerunt primum columnae crassitudinem altitudinis octaua parte, ut haberet speciem excelsiorem. Ita duobus discriminibus columnarum inuentionem unam uirili sine ornatu nudam specie, alteram muliebri substilitate, et ornatu simmetriaque sunt imitati. Posteri uero elegantia, subtilitateque iudiciorum progressi, et gracilioribus modulis delectati, septem crassitudinis diametros in altitudinem columnae Doricae, Ionicae octo semis constituerunt. Id autem genus, quod Iones fecerunt, primo Ionicum est nominatum.

"Tertium uero quod Corinthium dicitur, uirginalis habet gracilitatis imitationem, quod uirgines propter aetatis teneritatem gracilioribus membris figuratae, effectus recipiunt in ornatu uenustiores."

(*b*) *From "Methodus inveniendi lineas curvas, etc.," by Leonhard Euler, Bousquet, Lausanne and Geneva, 1744, p. 267.*

"Nisi ergo onus gestandum P majus sit quam $E(\pi\pi kk)/(aa)$, nulla prorsus incurvatio erit metuenda; contra verso si pondus P suerit majus, columna incurvationi resistere non poterit. Manente autem elasticitate columnae, atque adeo ejus crassitie eadem; pondus P, quod sine periculo gestare valet, erit reciproce ut quadratum altitudinis columnae: columnaque duplo altior quartam tantum oneris partem gestare poterit."

(*c*) *From "Résistance des pièces comprimées" par M. Considère, Congrès International des Procédés de Construction, Paris, 1889.*

"Pour déterminer le valeur de $\mathscr{E}$, il faut distinguer les fibres extérieures et intérieures par rapport à la courbure que prend la pièce comprimée.

"Pour les premières, la courbure détermine un allongement, et par suite une diminution de compression et l'on sait que dans ce cas, c'est-à-dire dans le retour vers l'état de repos, le métal se conduit comme s'il était complètement élastique. Pour les fibres extérieures, $\mathscr{E}$ est donc égale à E, coefficient d'élasticité parfaite.

"... et $\mathscr{E}$ ... sera égal à l'inclinaison de la tangente au point P par rapport à l'axe des r.

"Le moment total de flexion de la pièce comprimée étant produit par l'action simultanée des efforts des fibres convexes, et concaves, $\mathscr{E}$ doit donc ... avoir une valeur intermédiaire entre le constant E et le coefficient angulaire de la tangente à la courbe de déformation ..."

(*d*) *From the "Schweizerische Bauzeitung," Vol. 25, Nr. 25, p. 175, 22 June 1895.*

"Es ist natürlich immer möglich, die Grösse der Knickfestigkeit eines Stabes, in welchem die Elasticitätsgrenze überschritten ist, durch die Formel

$$P = 10\,TJ/L^2$$

auszudrücken, wo T irgendeine, uns unbekannte Funktion von den Stabdimensionen ist, aber diese Funktion ist nicht gleich tg, sonder grösser als diese. Die Gestalt dieser Funktion theoretisch zu finden, ist vorläufig unmöglich, es bleibt uns daher nur der empirische Weg. Aus den Versuchen von Tetmajer liesse sich der Wert von T natürlich leicht bestimmen, aber es wäre dann die Formel (3) nur eine überflüssige Komplikation der Tetmajer'schen."

St. Petersburg, 16th April 1895.
Adj. Prof. F. Jasinski

(*e*) *From "Über Knickfragen," by Fr. Engesser, Schweizerische Bauzeitung, Vol. 26, No. 4, p. 24, 27 July 1895.*

"Nach vorstehenden Gleichungen erscheint P streng genommen nicht von I, sondern von einer verwickelteren Querschnittsfunktion abhängig; Querschnitte gleichen Trägheitsmomentes aber verschiedenartiger Form ergeben daher ausserhalb der Elasticitätsgrenze nicht genau die gleiche Knickfestigkeit, wie dies in ähnlicher Weise auch bezüglich der Biegungsfestigkeit der Fall ist."

(*f*) *From "Untersuchungen über Knickfestigkeit," by Theodor von Kármán, Mitteilungen über Forschungsarbeiten auf dem Gebiete des Ingenieurwesens, p 18, Verein Deutscher Ingenieure, Heft 81, Berlin, 1910.*

"Aus diesen Figuren ist nun sowohl der Einfluss der Excentrizität auf die Höchstlast als der Verlauf des Vorganges nach der Ausknickung deutlich zu entnehmen. Namentlich sieht man, dass in Gegensatz zu dem elastischen Falle, wo die Kurven mit verschiedenen Excentrizitäten derselben Höchstlast asymptotisch zustreben, hier die Höchstlast schon durch sehr geringe Excentrizitäten erheblich vermindert wird.

"Dieser—in praktischer Hinsicht äusserst wichtige—Umstand steht damit in Zusammenhang, dass, während bei der elastischen Knickung die Ausbiegung unter fast unveränderlicher Last erfolgt, bei dem unelastischen Vorgange die Ausbiegung eine rasche Abnahme der Last

zur Folge hat—wie dies z.B. die Figur 18 klar zeigt . . . Später werden wir sehen, wie diese theoretischen Folgerungen durch die Versuche bestätigt wurden."

(g) *From "Untersuchungen über Knickfestigkeit," by Theodor von Kármán, Mitteilungen über Forschungsarbeiten auf dem Gebiete des Ingenieurwesens, p. 33, Verein Deutscher Ingenieure, Heft 81, Berlin, 1910.*

"Nachdem die Höchstlast erreicht wurde, trat bei den meisten Stäben eine ganz plötzliche Ausknickung ein, so dass Manometer und Rollenapparat einen kräftigen Ruck erhielten . . . Dies ist dadurch zu erklären, dass das Gleichgewicht im Gesamtsysteme Pumpe + Festigkeitsmaschine + Stab labil wird. Um sich den Vorgang klar zu machen, betrachte man das Kräftespiel in der Maschine in seiner Abhängigkeit von der Annäherung der beiden Schneidenlager. Die letztere ist gleich dem Kolbenwege vermindert um die Federung der Maschine, welche sich im wesentlichen aus der Streckung der Säulen und der Durchbiegung des Querbalkens zusammensetzt. Diese Federung kann als der Kraft proportional angenommen werden, welche auf den Stab ausgeübt wird. In Figur 29 sei auf x-Achse der Kolbenweg, auf die y-Achse die Kraft aufgetragen; die Zustände der Maschine, die je ein und derselben Wassermenge im Kolben entsprechen, werden dann durch parallele, schiefe Geraden ($\overline{BP}$) dargestellt. Anderseits entspricht den Verkürzungen des Probestabes ein bestimmter Kraftverlauf, der als Funktion dieser Verkürzung durch die Linie $\overline{AP}_k$ dargestellt ist. Die Schnittpunkte dieser Kurve mit den parallelen Geraden bestimmen die nacheinander folgenden Gleichgewichtszustände des Gesamtsystems. Da nun nach Überschreitung der Höchstlast P_k die Kurve rasch fällt, so kommt man zu einem Punkte P', wo sie durch eine der parallelen Geraden berührt wird. In diesem Punkte findet ein plötzlicher Gleichgewichtswechsel statt, indem die Last rasch zurückgeht, bis im Punkte P'' wieder eine stabile Gleichgewichtslage erreicht wird."

Appendix II

Perfectly Elastic Column Compressed in a Testing Machine

DERIVATION OF THE EQUATIONS

If an element of the pin-ended column shown in Fig. 3, bounded by two cross sections dx apart, is isolated, its equation of motion in the axial (x^*) direction is

$$(\partial^2 u/\partial t^2)=(E/\mu)(\partial^2 u/\partial x^{*2}) \quad . \quad . \quad (69)$$

where E is Young's modulus, μ the mass of a unit volume of the material, u the axial displacement of a section originally a distance x^* from the origin, and t is time. The transverse motion of the element is governed by the equation

$$(\partial^2/\partial x^{*2})\{EI[(\partial^2 y/\partial x^{*2})-(\partial^2 y_0/\partial x^{*2})]\}+ \\ +(\partial^2/\partial x^{*2})(Py)+\mu A(\partial^2 y/\partial t^2)=0 \quad (70)$$

where y is the transverse distance of the element from the straight line connecting the end points of the column, y_0 is the initial value of y at $t=0$ in the unstressed state, EI is the bending rigidity, A the cross-sectional area, and P the compressive force in the particular section of the column*. It is assumed that the minor principal axis of inertia of the cross section is perpendicular to the x^*-y plane. Since

$$P=-EA(\partial u/\partial x^*) \quad . \quad . \quad (71)$$

equations (69) and (70) are connected by equation (71).

Solution of these simultaneous equations with prescribed initial and boundary conditions is difficult. Fortunately the pressure waves caused by end displacements travel much faster along the column than the speed with which the lateral displacements take place. Hence it is permissible to consider the axial compressive force P a constant along the column. This means that equations (69) and (71) can be replaced by an integral expression which correlates the compressive load with the displacement ct of the loading head at time t and with the change in the distance between the end points of the column. When the cross section of the column is constant, the expression is

$$P=(EA/L)\{ct-(1/2)\int_0^L[(\partial y/\partial x^*)^2-(\partial y_0/\partial x^*)^2]\,dx^*\} \quad (72)$$

It will now be assumed that the bending rigidity EI is constant and the initial shape is described by the equation

$$y_0=e\rho\sin(\pi x^*/L) \quad . \quad . \quad (73)$$

The initial velocity $\partial y/\partial t$ is assumed to be zero. Under these conditions the assumption†

$$y=F\sin(\pi x^*/L) \quad . \quad . \quad . \quad (74)$$

where F is an unknown function of time, satisfies the equation of motion provided

$$f''+\Omega[(1-\xi)f-e+(1/4)f^3-(1/4)e^2 f]=0 \quad . \quad (75)$$

In this equation the following non-dimensional quantities appear:

$$f=F/\rho \qquad \xi=(1/\varepsilon_E)(ct/L) \\ \varepsilon_E=\pi^2(\rho/L)^2 \qquad \Omega=\pi^2\varepsilon_E{}^3(E/\mu c^2) \quad . \quad (76)$$

Ω is a dynamic similarity number, ρ is the radius of gyration of the cross section, and the primes indicate differentiation with respect to the non-dimensional time parameter ξ. Equation (72) can now be re-written if use is made of equations (73), (74) and (76):

$$(P/P_E)=\xi-(1/4)(f^2-e^2) \quad . \quad . \quad (77)$$

where

$$P_E=EA\varepsilon_E. \quad . \quad . \quad . \quad (78)$$

* For the derivation of these equations *see*, for instance, Lord Rayleigh's *The Theory of Sound* [98]

† For details of this analysis *see* Reference 22.

STATIC SOLUTION

When the acceleration term in equation (75) is disregarded, the static solution f_{st} can be expressed implicitly in the form

$$\xi = 1 - (e/f_{st}) + (1/4)(f_{st}^2 - e^2) \quad . \quad . \quad (79)$$

The corresponding axial load is from equation (77)

$$(P/P_E)_{st} = 1 - (e/f_{st}) \quad . \quad . \quad (80)$$

PERTURBATION EQUATION

It is conjectured that at the rates of loading usual in testing columns, the dynamic solution of equation (75) does not differ much from the static solution. Under these conditions the solution can be obtained in a manner suggested to the author by Professor George F. Carrier of Harvard University; a description of the method of asymptotic solution which is used in the analysis may be found, for instance, in Reference 99.

One may set

$$f = f_{st} + \eta \quad . \quad . \quad . \quad (81)$$

substitute this expression in equation (75) and consider η small as compared to f_{st}. The result is a differential equation for η:

$$\eta'' + \Omega\,[1 - \xi + (1/4)(3f_{st}^2 - e^2)]\,\eta = -f_{st}'' \quad . \quad (82)$$

It will be shown later that f_{st}'' is small as compared to η''. Equation (82) can therefore be replaced in a satisfactory approximation by

$$\eta'' + h^2\chi\eta = 0 \quad . \quad . \quad . \quad (83)$$

where the new symbols are defined as

$$h^2 = \Omega \qquad \chi = 1 - \xi + (1/4)(3f_{st}^2 - e^2) \quad . \quad (84)$$

ASYMPTOTIC SOLUTION

In the ordinary column test Ω is a large number; for steel and aluminium alloys $(E/\mu)^{\frac{1}{2}}$ is about 200,000 in./sec. and thus a slenderness ratio (L/ρ) of 100 and a loading head velocity c of 2×10^{-3} in./sec. result in an Ω value of 10^8. The value of h is then 10^4. With such high values of h it is advantageous to expand the solution in the series

$$\eta = \psi\,(\exp h\omega)\,[1 + (f_1/h) + (f_2/h^2) + \ldots] \quad . \quad (85)$$

where ψ, ω, f_1, f_2, ... are functions of ξ. The second derivative of η with respect to ξ is

$$\eta'' = (\exp h\omega)\,[\psi\omega'^2 h^2 + (2\psi'\omega' + \psi\omega'' + \psi\omega'^2 f_1)\,h + 0\,(1)] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

If the expressions presented in equations (85) and (86) are substituted in equation (83) and the sum of the terms multiplied by h^2 is equated to zero one obtains

$$\psi\omega'^2 + \psi\chi = 0 \quad . \quad . \quad . \quad (87)$$

Similarly the terms multiplied by h yield

$$2\psi'\omega' + \psi\omega'' + \psi\omega'^2 f_1 + \chi\psi f_1 = 0 \quad . \quad (88)$$

Because of equation (87) one can write equation (88) in the form

$$2\psi'\omega' + \psi\omega'' = 0 \quad . \quad . \quad . \quad (89)$$

From equation (87)

$$\omega'^2 = -\chi \quad . \quad . \quad . \quad (90)$$

and from equation (89)

$$(\psi'/\psi) = -(1/2)(\omega''/\omega') \quad . \quad . \quad (91)$$

If equation (90) is differentiated with respect to ξ and the resulting equation is solved for ω''

$$\omega'' = -(1/2)(\chi'/\omega') \quad . \quad . \quad (92)$$

Substitution in equation (91) and consideration of equation (90) yield

$$(\psi'/\psi) = -(1/4)(\chi'/\chi) \quad . \quad . \quad (93)$$

After integration this becomes

$$\psi = C^*\chi^{-\frac{1}{4}} \quad . \quad . \quad . \quad (94)$$

where C^* is a constant of integration. If the first term alone is retained from the bracketed expression in equation (85), one has

$$\eta = C^*\chi^{-\frac{1}{4}} \exp\left[\pm ih \int_{\xi_0}^{\xi} \chi^{\frac{1}{2}}\, d\xi\right] \quad . \quad (95)$$

When the integral is real, this can also be written as

$$\eta = C^*\chi^{-\frac{1}{4}} \sin(h\chi^* - \varphi^*) \quad . \quad . \quad (96)$$

where

$$\chi^* = \int_{\xi_0}^{\xi} \chi^{\frac{1}{2}}\, d\xi \quad . \quad . \quad . \quad (97)$$

and φ^* is a phase angle depending on the initial conditions. The dominant term in the expression for the velocity of the perturbation oscillations is

$$d\eta/dt = (c/L\varepsilon_E)(d\eta/d\xi) = C^*(c/L\varepsilon_E)\,h\chi^{\frac{1}{4}} \cos(h\chi^* - \varphi^*) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

THE MAGNITUDE OF THE OSCILLATIONS

According to equations (96) and (97) the history of the perturbation oscillations is determined by the function χ. If $1 - \xi$ is calculated from equation (79) and substituted in equation (84) one obtains

$$\chi = (2e + f_{st}^3)/2f_{st} \quad . \quad . \quad . \quad (99)$$

If the testing machine is started when there is no load acting on the column, and ξ is measured from that instant, f_{st} is equal to e; also

$$\chi = 1 + (e^2/2) \qquad (\xi = 0) \quad . \quad . \quad (100)$$

which is slightly greater than unity. The differential coefficient of f_{st} can be calculated from equation (79):

$$(df_{st}/d\xi) = 2f_{st}^2/(2e + f_{st}^3) \quad . \quad . \quad (101)$$

With the aid of this expression the differential coefficient of χ can be obtained from equation (99):

$$(d\chi/d\xi) = 2\,(f_{st}^3 - e)/(f_{st}^3 + 2e) \quad . \quad . \quad (102)$$

When the differential coefficient vanishes, χ has an extremum. This happens when

$$f_{st}^3 = e \qquad (\chi = \chi_{extr}) \quad . \quad . \quad (103)$$

As the second differential coefficient of χ

$$(d^2\chi/d\xi^2) = 36ef_{st}^4/(2e + f_{st}^3)^3 \quad . \quad . \quad (104)$$

is always positive, the curve obtained by plotting χ against ξ is everywhere concave upward and the

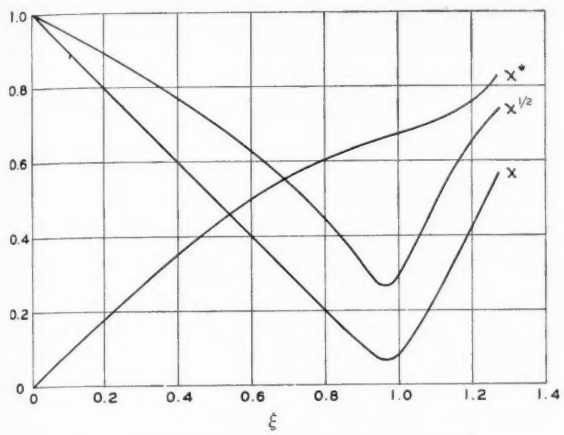

FIGURE 48. Variation of χ, $\chi^{1/2}$, and χ^* with ξ.

extremum of χ just calculated is a minimum. Its value is (*see* Fig. 48)

$$\chi_{\min}=(3/2)\,e^{2/3} \quad . \quad . \quad . \quad (105)$$

The corresponding value of ξ is, from equation (79)

$$\xi=1-(3/4)\,e^{2/3}-(1/4)\,e^2 \qquad (\chi=\chi_{\min}) \quad . \quad (106)$$

These equations permit an evaluation of the growth of the perturbation oscillations. Naturally the maximum amplitude is reached when χ is a minimum; hence the maximum value of the magnification R_a can be calculated from equations (100), (105) and (96) as

$$R_a=(2/3)^{1/4}\,[1+(e^2/2)]^{1/4}\,e^{-1/6} \quad . \quad (107)$$

Since $(e^2/2)$ is small as compared to unity in all column tests, equation (107) is in good approximation

$$R_a\simeq 0{\cdot}9e^{-1/6} \quad . \quad . \quad . \quad (108)$$

For a very well centred column the value of e may be as low as 10^{-3}; then the magnification is

$$R_a\simeq 2{\cdot}85 \quad . \quad . \quad . \quad (109)$$

The non-dimensional time elapsed between two consecutive maxima of the oscillations can also be calculated by evaluating the integral of equation (97).* Since $d\xi/df_{st}$ is a monotonic function, the role of dependent and independent variable may be interchanged. In this manner χ^* can be evaluated as a function of f_{st}, and indirectly, as a function of ξ. We may write

$$\chi^*=\int_e^{f_{st}}\chi^{1/2}\,(d\xi/df_{st})\,df_{st}$$

$$=(1/2^{3/2})\int_e^{f_{st}}(1/f_{st})^{5/2}\,(2e+f_{st}{}^3)^{3/2}\,df_{st}$$

$$. \quad . \quad . \quad . \quad . \quad . \quad (110)$$

If the notation is introduced

$$z=f_{st}{}^3/2e \quad \text{or} \quad f_{st}=(2ez)^{1/3} \quad . \quad . \quad (111)$$

and it is observed that

$$f_{st}=e \quad \text{when} \quad z=e^2/2 \quad . \quad . \quad (112)$$

substitution in equation (110) yields

$$\chi^*=(1/3)\,(e/2^{1/2})\int_{e^2/2}^{z}[(1+z)/z]^{3/2}\,dz \quad . \quad (113)$$

The indefinite integral is

$$\int[(1+z)/z]^{3/2}\,dz=(z-2)\,[(1+z)/z]^{1/2}+$$
$$+3\log\,[z^{1/2}+(1+z)^{1/2}] \quad . \quad . \quad (114)$$

At the lower limit, where $z=e^2/2$, the value of this integral is

$$[(e^2/2)-2]\,[1+(2/e^2)]^{1/2}+3\log\{(e/2^{1/2})+[1+(e^2/2)]^{1/2}\}$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

Since e is always small, one may set

$$(e^2/2)-2\sim-2 \qquad (e/2^{1/2})+[1+(e^2/2)]^{1/2}\sim 1$$
$$1+(2/e^2)\sim 2/e^2 \quad . \quad . \quad . \quad (116)$$

Thus the approximate value of the integral at the lower limit is

$$(-2)\,(2^{1/2}/e)+3\log\,(1)=-(2)^{3/2}/e$$

since $\log\,(1)=0$. Consequently

$$\chi^*=(2/3)+(1/3)\,(e/2^{1/2})$$
$$\{(z-2)\,[(1+z)/z]^{1/2}+3\log\,[z^{1/2}+(1+z)^{1/2}]\}$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

This can also be expressed in terms of f_{st}:

$$\chi^*=(2/3)+(1/3)\,(e/2^{1/2})\Big(\,[(f_{st}{}^3/2e)-2]$$
$$[1+(2e/f_{st}{}^3)]^{1/2}+3\log\{[f_{st}{}^3/2e]^{1/2}+[1+(f_{st}{}^3/2e)]^{1/2}\}\Big)$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

In addition one has

$$\xi=1-(e/f_{st})+(1/4)\,(f_{st}{}^2-e^2) \quad . \quad . \quad (119)$$

If e is given, a value of f_{st} can be selected and the corresponding values of χ^* and ξ can be computed. This was done for $e=0{\cdot}01$ and the results were plotted in Fig. 48. It is noted that the minimum of χ occurs at $(f_{st}/e)=21{\cdot}6$.

As in the neighbourhood of $\chi_{\min}$ (and of $\xi=1$) the value of $(d\chi^*/d\xi)$ is about 0·3, it follows from equation (96) that the non-dimensional time X elapsed between consecutive maxima of the oscillations near buckling is

$$X=2\pi/0{\cdot}3\,h\sim 20/h \quad . \quad . \quad (120)$$

The time T itself is then

$$T=(\varepsilon_E L/c)\,X \quad . \quad . \quad . \quad (121)$$

When $h=10^4$, equation (120) yields $X=2\times10^{-3}$. If it is assumed that $L=10$ in., $\varepsilon_E=10^{-3}$, and $c=2\times10^{-3}$ in./sec., equation (121) gives

$$T=0{\cdot}01 \text{ sec.} \quad . \quad . \quad . \quad (122)$$

This is of course not a true period because the interval between consecutive maxima changes slowly with time.

On the basis of the investigation presented, it can be concluded that the motion of the perfectly elastic column in the testing machine is the motion obtainable from considerations of static equilibrium upon which a

* This integration was performed by Professor F. V. Pohle.

rapid oscillation is superimposed. The frequency of this oscillation is high and it changes slowly during the loading process; the amplitude increases from the start of the process until the Euler displacement $\xi=1$ is reached approximately, and then begins to decrease, but the amplitude ratio is not likely to exceed 3 under ordinary conditions of testing.

PROOF THAT f_{st}'' IS SMALL AS COMPARED TO η''

The second differential coefficient of f_{st} is

$$(d^2 f_{st}/\partial\xi^2)=4f_{st}^3(4e-f_{st}^3)/(2e+f_{st}^3)^3 \quad . \quad (123)$$

It is an extremum when

$$f_{st}^6-12ef_{st}^3+8e^2=0 \quad . \quad . \quad (124)$$

The first extremum, a maximum, occurs when

$$f_{st}=0{\cdot}891\,e^{1/3} \quad . \quad . \quad . \quad (125)$$

Its value is

$$(f_{st}'')_{max}=0{\cdot}469\,e^{-1} \quad . \quad . \quad (126)$$

The corresponding value of ξ is

$$\xi=1-0{\cdot}924\,e^{2/3}-(1/4)e^2 \quad . \quad . \quad (127)$$

which is slightly less than unity for the values of e usual in column testing. The second extremum is in the range of ξ which is of no practical interest in the investigation of the routine column test.

The maximum of f_{st}'' must be compared with the minimum of η''. Equation (96) can be differentiated with respect to ξ to give

$$\eta''=-C^*h^2\chi^{3/4}\sin(h\chi^*-\varphi^*)+O(h) \quad . \quad (128)$$

The symbol $O(h)$ designates terms containing h, χ, χ', and χ''. As the maximum of the latter is

$$(\chi'')_{max}=1{\cdot}45\,e^{-2/3} \quad . \quad . \quad (129)$$

when

$$f_{st}^3=(8/5)e \quad . \quad . \quad . \quad (130)$$

and the maximum values of χ and of its first derivatives are approximately unity, the terms represented by $O(h)$ can be disregarded in equation (128) when the order of magnitude of η'' is investigated for the routine column test.

The minimum of χ was determined earlier. From equation (105)

$$(\chi^{3/4})_{min}=(3/2)^{3/4}\,e^{1/2} \quad . \quad . \quad (131)$$

Hence the minimum of the amplitude of η'' is

$$1{\cdot}306\,h^2e^{1/2}C^*$$

If the initial disturbance is of the order of magnitude of e, the requirement that η'' be always large as compared to f_{st}'' is

$$1{\cdot}306\,h^2\,e^{3/2}\gg 0{\cdot}469\,e^{-1} \quad . \quad . \quad (132)$$

This can also be written in the form

$$2{\cdot}78\,h^2\,e^{5/2}\gg 1 \quad . \quad . \quad . \quad (133)$$

The inequality is satisfied when $e=10^{-3}$ and $h^2\geqslant 10^9$; and when $e=10^{-2}$ and $h\geqslant 10^6$.

VALIDITY OF THE ASYMPTOTIC SOLUTION

If the asymptotic solution

$$\eta=\chi^{-1/4}\exp(ih\chi^*) \qquad \chi^*=\int_{\xi_0}^{\xi}\chi^{1/2}\,d\xi$$

is differentiated twice with respect to ξ one obtains

$$\eta''=[-(1/4)\chi^{-5/4}\chi''+(5/16)\chi^{-9/4}\chi'^2-\\ -h^2\chi^{3/4}]\exp(ih\chi^*) \quad . \quad (134)$$

Comparison with equation (83) shows that the solution represents a good approximation if the first two terms in the brackets are small as compared to the third term. Hence the condition of the validity of the solution is

$$(1/4)(\chi''/\chi^2)-(5/16)(\chi'^2/\chi^3)\ll h^2 \quad . \quad (135)$$

The functions χ, χ', and χ'' were given in equations (99), (102), and (104), respectively. Substitutions yield

$$(1/4)(\chi''/\chi^2)=36\,eq^2/(q+2e)^5 \quad . \quad (136)$$

with

$$q=f_{st}^3. \quad . \quad . \quad . \quad (137)$$

Similarly

$$(5/16)(\chi'^2/\chi^3)=10q(e-q)^2/(q+2e)^5 \quad . \quad (138)$$

As

$$(d/dq)[(1/4)(\chi''/\chi^2)]=36e(4eq-3q^2)/(q+2e)^6 \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

and

$$(d/dq)[(5/16)(\chi'^2/\chi^3)] \\ =20(e-q)(q^2-5eq+e^2)/(q+2e)^6 \quad . \quad (140)$$

the condition for an extremum of the left hand member of inequality (135) is

$$(q/e)^3-11{\cdot}4(q/e)^2+13{\cdot}2(q/e)-1=0 \quad . \quad (141)$$

The roots of this cubic are

$$(q_1/e)=\ 0{\cdot}081446$$
$$(q_2/e)=\ 1{\cdot}215263$$
$$(q_3/e)=10{\cdot}103292 \quad . \quad . \quad (142)$$

The corresponding extrema are

$$q_1=-0{\cdot}0115\,e^{-2}$$
$$q_2=+0{\cdot}15309\,e^{-2}$$
$$q_3=-0{\cdot}0178\,e^{-2} \quad . \quad . \quad . \quad (143)$$

In view of these equations it is certain that the left hand member of inequality (135) is not greater than $0{\cdot}15309\,e^{-2}$. Hence the inequality is satisfied if

$$h^2\gg 0{\cdot}15309\,e^{-2} \quad . \quad . \quad . \quad (144)$$

When $e=10^{-2}$, h^2 must exceed 10^4, and when $e=10^{-3}$, h^2 must exceed 10^6. These conditions are always satisfied in a routine column test.

Appendix III

Non-linear Column Compressed in a Testing Machine

THE STRESS-STRAIN LAW

The simplest non-linear stress-strain law that corresponds roughly to the actual behaviour of the material is the power law

$$(\varepsilon/\varepsilon_0)=(\sigma/\sigma_0)^n \quad . \quad . \quad . \quad (145)$$

where n is a positive number greater than unity. If n is not an odd integer, tensile and compressive stresses must be treated separately; when it is an odd integer, equation (145) is valid for compression as well as for tension. Tensile stresses and strains will be considered positive and compressive stresses and strains negative. Curves computed from this law for different odd integral values of n are shown in Fig. 10.

Much better agreement between assumption and experiment can be had if the relationship selected is a polynomial. Ramberg and Osgood[90] have shown that the equation

$$(\varepsilon/\varepsilon_0)=(\sigma/\sigma_0)+\alpha^*(\sigma/\sigma_0)^n \quad . \quad . \quad (146)$$

is capable of representing the stress-strain relationship of all structural metals with a satisfactory accuracy for engineering purposes.

Equation (145) can also be written in the form

$$\varepsilon=(\sigma/K)^n. \quad . \quad . \quad . \quad (147)$$

Differentiation with respect to ε yields

$$1=(n/K)(\sigma/K)^{n-1}(d\sigma/d\varepsilon)$$

from which one obtains

$$E_t=(d\sigma/d\varepsilon)=(K/n)/(\sigma/K)^{n-1}=(K^n/n)/\sigma^{n-1}$$
$$=(K/n)/\varepsilon^{(n-1)/n} \quad . \quad . \quad (148)$$

where E_t is known as the tangent modulus of the material. As at any value of the stress σ the resistance of the material to additional infinitesimal deformations is characterised by the tangent modulus, the critical stress of the column can be calculated from the Euler formula if, in it, Young's modulus E is replaced by the tangent modulus E_t:

$$\sigma_{cr}=\pi^2E_t/(L/\rho)^2 \quad . \quad . \quad . \quad (149)$$

where ρ is the radius of gyration of the cross section of the column and (L/ρ) is the slenderness ratio of the column. Substitution from equation (148) and solution give

$$\sigma_{cr}=K(\pi^2/n)^{1/n}/(L/\rho)^{2/n}. \quad . \quad . \quad (150)$$

Further manipulations yield

$$\varepsilon_{cr}=\varepsilon_E/n \quad . \quad . \quad . \quad (151)$$

where ε_E is the critical value of the uniform strain at which the straight-line configuration of a compressed, initially straight, column becomes unstable according to Euler:

$$\varepsilon_E=\pi^2/(L/\rho)^2. \quad . \quad . \quad . \quad (152)$$

COLUMN OF IDEALISED I-SECTION WHOSE MATERIAL FOLLOWS A CUBIC LAW

The calculations next to be presented will be done for a material for which $n=3$. With K arbitrarily chosen as 300,000 lb./in.^{2} the stress-strain law is

$$\varepsilon=(\sigma/300{,}000)^3. \quad . \quad . \quad . \quad (153)$$

From equation (148) the tangent modulus is

$$E_t=100{,}000/(\sigma/300{,}000)^2. \quad . \quad . \quad (154)$$

These relationships are illustrated in Figs. 11 and 12.

Moreover, the cross section of the column is assumed as shown in Fig. 14. It consists of two concentrated areas $A/2$ of material as the flanges which are kept a distance h^* apart by a web of vanishingly small area but of infinite shearing rigidity.

The radius of gyration ρ of this section is

$$\rho=(h^*/2). \quad . \quad . \quad . \quad (155)$$

Thus the critical stress becomes

$$\sigma_{cr}=447{,}000/(L/\rho)^{2/3}=281{,}000/(L/h^*)^{2/3} \quad (156)$$

and the critical strain is

$$\varepsilon_{cr}=(\varepsilon_E/3). \quad . \quad . \quad . \quad (157)$$

The critical stress is plotted against the slenderness ratio in Fig. 13.

DERIVATION OF THE DIFFERENTIAL EQUATION

When the idealised I-section column of Fig. 14 is simultaneously compressed and bent, the strain on the concave side can be denoted as ε_c and that on the convex side as ε_t. The elongation of an infinitesimal element ds of the column is then $\varepsilon_c ds$ and $\varepsilon_t ds$ in the two flanges (*see* equation (145)). The difference in elongation $(\varepsilon_t-\varepsilon_c)ds$ divided by h^*, the distance between the flanges, is the relative angle of rotation $d\alpha^{**}$ of two sections originally ds apart. But this distance is increased from ds to $ds[1+(1/2)(\varepsilon_t+\varepsilon_c)]$ and thus the angle $d\alpha^{**}$ can also be expressed as

$$d\alpha^{**}=ds[1+(1/2)(\varepsilon_t+\varepsilon_c)]/r$$

where r is the radius of curvature of the axis of the column. As in all the ensuing calculations the average extension is small as compared to unity, we can write

$$d\alpha^{**}=ds/r.$$

If this equation is compared with the expression found earlier for $d\alpha$, namely

$$d\alpha^{**}=(\varepsilon_t-\varepsilon_c)(ds/h^*)$$

the curvature $(1/r)$ can be given as

$$(1/r)=(\varepsilon_t-\varepsilon_c)/h^*. \quad . \quad . \quad . \quad (158)$$

The stresses in the flanges can be found with the aid of equation (147). If both members of the equation are raised to the $(1/n)^{th}$ power, one can solve for σ:

$$\sigma=K\varepsilon^{1/n}. \quad . \quad . \quad . \quad (159)$$

Thus the axial force in the column (positive when tensile) is

$$P=(1/2)\,AK\,(\varepsilon_t^{1/n}+\varepsilon_c^{1/n}) \quad . \quad . \quad (160)$$

and the moment of the forces with respect to the axis of the column is

$$M=(1/4)\,AKh^*(\varepsilon_t^{1/n}-\varepsilon_c^{1/n}). \quad . \quad . \quad (161)$$

The following non-dimensional force and moment quantities will now be introduced:

$$p=(2P/AK)=\varepsilon_t^{1/n}+\varepsilon_c^{1/n} \quad . \quad . \quad (162)$$

$$m=(4M/AKh^*)=\varepsilon_t^{1/n}-\varepsilon_c^{1/n}. \quad . \quad . \quad (163)$$

Solution for ε_t and ε_c yields

$$\varepsilon_t=(1/2)^n(p+m)^n \quad . \quad . \quad . \quad (164)$$

$$\varepsilon_c=(1/2)^n(p-m)^n. \quad . \quad . \quad (165)$$

When the lateral deflections y of the column are small, the curvature $1/r$ is equal to the second derivative of y with respect to the axial co-ordinate x^*; a negative sign is used with this curvature in order to obtain agreement with the sign convention adopted for deflections and moments:

$$(1/r)=-d^2y/dx^{*2}.$$

The non-dimensional lateral deflection w is defined as

$$w=y/\rho=2y/h^*. \quad . \quad . \quad . \quad (166)$$

and the non-dimensional axial co-ordinate x as

$$x=x^*/L.$$

Therefore

$$1/r=-d^2y/dx^{*2}=-(h^*/2L^2)(d^2w/dx^2)$$

$$=-(h^*/2L^2)\,w''. \quad . \quad . \quad . \quad (167)$$

From equation (158)

$$-[(h^*/2)/L^2]w''=(1/h^*)(\varepsilon_t-\varepsilon_c)$$

and with the aid of equations (164) and (165) one has

$$-[(h^*/2)/L^2]\,w''=(1/h^*)(1/2)^n[(p+m)^n-(p-m)^n].$$

This can also be written as

$$w''=-2\,(L/h^*)^2(1/2)^n[(p+m)^n-(p-m)^n]. \quad (168)$$

A differential equation in terms of deflections can be obtained from equation (168) if p and m are expressed in terms of w.

INITIALLY STRAIGHT COLUMN

When the column is perfectly straight and centrally loaded in its initial state, the bending moment in any of its sections is

$$M=-Py \quad . \quad . \quad . \quad (169)$$

or in the non-dimensional quantities

$$m=-pw. \quad . \quad . \quad . \quad (170)$$

Substitution in equation (168) yields

$$w''=-2\,(L/h^*)^2(1/2)^n p^n[(1-w)^n-(1+w)^n]$$

which can also be written in the form

$$w''=-\Pi\,[(1-w)^n-(1+w)^n] \quad . \quad (171)$$

where $\Pi=2\,(L/h^*)^2(p/2)^n=(\pi^2/2)(\varepsilon_{nom}/\varepsilon_E)$. (172)

and ε_{nom} is the strain corresponding to the average stress

$$\sigma_{ave}=P/A=(1/2)\,pK. \quad . \quad . \quad (173)$$

The nominal strain is positive when P is a tensile force. When $n=3$, equation (171) becomes

$$w''=2\Pi\,(3w+w^3). \quad . \quad . \quad . \quad (174)$$

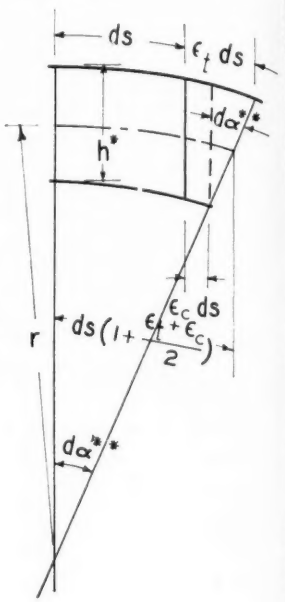

FIGURE 49. Geometry of strain in column.

SOLUTION FOR $n=3$

The solution is assumed in the form

$$w=\sum_{i=1}^{\infty} a_i \sin i\gamma \qquad \gamma=\pi x^*/L=\pi x \quad . \quad (175)$$

which satisfies the boundary conditions

$$w=w''=0 \quad \text{when} \quad \gamma=0,\pi. \quad . \quad . \quad (176)$$

From equation (175) it follows that

$$w''=-\pi^2\sum_{i=1}^{\infty} i^2 a_i \sin i\gamma \quad . \quad . \quad (177)$$

$$w^3=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_i a_j a_k \sin i\gamma \sin j\gamma \sin k\gamma \quad . \quad (178)$$

and

$$\sin i\gamma \sin j\gamma \sin k\gamma=(1/4)\,[\sin(i+j-k)\gamma+$$
$$+\sin(j+k-i)\gamma+\sin(k+i-j)\gamma-\sin(i+j+k)\gamma].$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (178a)$$

When only the first three terms of the series are retained one has

$$4w^3=a_1^3(3\sin\gamma-\sin 3\gamma)+3a_1^2a_2(2\sin 2\gamma-\sin 4\gamma)+$$
$$+3a_1^2a_3(2\sin 3\gamma-\sin 5\gamma-\sin\gamma)+3a_2^2a_1(\sin 3\gamma+$$
$$+2\sin\gamma-\sin 5\gamma)+3a_2^2a_3(\sin\gamma+2\sin 3\gamma-\sin 7\gamma)+$$
$$+3a_3^2a_1(\sin 5\gamma+2\sin\gamma-\sin 7\gamma)+3a_3^2a_2(\sin 4\gamma+$$
$$+2\sin 2\gamma-\sin 8\gamma)+a_2^3(3\sin 2\gamma-\sin 6\gamma)+$$
$$+a_3^3(3\sin 3\gamma-\sin 9\gamma)+6a_1a_2a_3(\sin 4\gamma+\sin 2\gamma-$$
$$-\sin 6\gamma). \quad . \quad . \quad . \quad . \quad (179)$$

With only three terms available, the number of conditions of equilibrium that can be satisfied is three. Hence in the equilibrium equations only the first three components of the nine series are needed:

$$4w^3=(3a_1^3-3a_1^2a_3+6a_2^2a_1+3a_2^2a_3+6a_3^2a_1)\sin\gamma+$$
$$+(6a_1^2a_2+6a_3^2a_2+6a_1a_2a_3+3a_2^3)\sin 2\gamma+$$
$$+(-a_1^3+6a_1^2a_3+3a_2^2a_1+6a_2^2a_3+3a_3^3)\sin 3\gamma. \quad . \quad (180)$$

Finally, if the second term is excluded for reasons of symmetry

$$4w^3=(3a_1^3-3a_1^2a_3+6a_1a_3^2)\sin\gamma+$$
$$+(-a_1^3+6a_1^2a_3+3a_3^3)\sin 3\gamma. \quad . \quad (181)$$

If upon substitution from equations (175), (177) and (181) into equation (174) the coefficients of the multipliers of $\sin\gamma$ and $\sin 3\gamma$ are set equal to zero individually, the following two equations are obtained:

$$(2\pi^2+12\Pi)a_1+3\Pi a_1^3+\Pi(6a_1a_3^2-3a_1^2a_3)=0 \quad (182)$$

$$(18\pi^2+12\Pi)a_3+3\Pi a_3^3+\Pi(6a_1^2a_3-a_1^3)=0. \quad (183)$$

In a first approximation the deflected shape is assumed as a single half sine wave and a_3 is set equal to zero. The resulting cubic has two kinds of solutions; either $a_1=0$ or

$$2\pi^2+12\Pi+3\Pi a_1^2=0. \quad . \quad . \quad (184)$$

It follows then that

$$\Pi=-2\pi^2/(12+3a_1^2). \quad . \quad . \quad (185)$$

When $3a_1^2$ is negligibly small as compared to 12, one obtains

$$\Pi=\Pi_{cr}=-\pi^2/6. \quad . \quad . \quad . \quad (185a)$$

In view of the definition of Π in equation (172), the above equation is equivalent to

$$(\varepsilon_{nom}/\varepsilon_E)=-(1/3)$$

which agrees with equation (157).

The connection between load and amplitude of deflections can be easily computed from equation (185). It is worth noting that the curve representing this relationship begins with a horizontal tangent as can be seen from the equation

$$(d\Pi/da_1)=12\pi^2a_1/(12+3a_1^2)^2 \quad . \quad (186)$$

which vanishes when $a_1=0$.

The relationship between load and amplitude is shown in Fig. 15. Values of a_1 were assumed first and the corresponding values of Π were calculated. For very small values of a_1 equilibrium is possible if

$$\Pi=\Pi_{cr}=-\pi^2/6 \quad \text{and} \quad (\varepsilon_{nom}/\varepsilon_E)=-(1/3)$$

as was stated earlier. As a_1 increases, the absolute value of Π decreases but at all times

$$\Pi/\Pi_{cr}=(\varepsilon_{nom}/\varepsilon_{ave,cr}). \quad . \quad . \quad (187)$$

It follows then from equation (147) with $n=3$ that

$$\Pi/\Pi_{cr}=(\sigma_{ave}/\sigma_{ave,cr})^3. \quad . \quad . \quad (188)$$

When both a_1 and a_3 are considered, the following step-by-step approximation procedure can be used: First, equation (182) can be solved for Π:

$$\Pi=-2\pi^2/(12+3a_1^2+6a_3^2-3a_1a_3). \quad . \quad (189)$$

In equation (183) the term containing a_3^2 is neglected and the equation is solved for a_3:

$$a_3=a_1^3/[(18\pi^2/\Pi)+12+6a_1^2]. \quad . \quad (190)$$

In the computations a_1 is assumed and Π is calculated from equation (185). Next the assumed value of a_1 and the computed value of Π are introduced in equation (190) and a_3 is obtained. With this a_3 and the assumed value of a_1, a new value is computed for Π from equation (189). Next equation (190) is used to obtain an improved value of a_3 and the process is continued until the difference between assumed and calculated values for Π and a_3 becomes satisfactorily small.

When a_3 is very small, the value of Π is little influenced by it. Hence Π can be introduced from equation (185) into equation (190). The approximate formula is obtained:

$$a_3=-a_1^3/(96+21a_1^2). \quad . \quad . \quad (191)$$

The values of a_3 are plotted in Fig. 16. The conclusion can be drawn from the curve that the shape of the deflected column is almost exactly the simple sine-shape characteristic of perfectly elastic columns. Finally it was checked whether it was permissible to omit the term containing a_3^3 from equation (183). At $a_1=5$ the effect of this term was smaller than the errors inherent in computations with the slide rule.

INITIALLY SLIGHTLY CURVED COLUMN

The column will now be assumed to have a slight initial curvature according to the formula

$$y_0=e\rho\sin(\pi x^*/L). \quad . \quad . \quad (192)$$

In non-dimensional units

$$w_0=e\sin\pi x \quad . \quad . \quad . \quad (193)$$

with $w_0=y_0/\rho$ and $x=x^*/L$. . . (194)

If w_{ad} designates the non-dimensional additional, not the total deflections, equation (168) becomes:

$$w_{ad}''=-2(L/h^*)^2(1/2)^n[(p+m)^n-(p-m)^n].$$

However, the bending moment in a section of the initially curved column is

$$M=-P(y_{ad}+y_0) \quad . \quad . \quad (195)$$

and thus $m=-p(w_{ad}+w_0)$. . . (195a)

Substitution in equation (168) yields

$$w_{ad}''=-\Pi\{[1-(w_{ad}+w_0)]^n-[1+(w_{ad}+w_0)]^n\}.$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (196)$$

If now the total non-dimensional deflection is designated as w,

$$w=w_{ad}+w_0$$

equation (196) becomes

$$w''-w_0''=-\Pi[(1-w)^n-(1+w)^n] \quad . \quad (197)$$

and finally, with $n=3$,

$$w''-2\Pi(3w+w^3)=w_0''. \quad . \quad . \quad (198)$$

As it was found that the higher harmonics are unimportant, the analysis will be restricted to the first

mode. The non-dimensional total displacement is assumed as

$$w = a \sin \pi x. \quad . \quad . \quad . \quad (199)$$

Substitution of the quantities defined in equations (194) and (199) in equation (198) leads to

$$-\pi^2 a - 2\Pi [3a + (3/4) a^3] = -\pi^2 e.$$

Solution for Π gives

$$\Pi = -(2\pi^2/3)[1 - (e/a)]/(4 + a^2).$$

As $\Pi_{cr} = -(\pi^2/6)$, one has

$$\Pi/\Pi_{cr} = [1 - (e/a)]/[1 + (a/2)^2]. \quad . \quad (200)$$

From this equation (Π/Π_{cr}) can be calculated for given values of e and a. The average stress acting on the column and the load follow then from the relationships

$$(P/P_{cr}) = (\sigma_{ave}/\sigma_{ave,0}) = (\Pi/\Pi_{cr})^{1/3}. \quad . \quad (201)$$

Values of (P/P_{cr}) were computed for $e = 1$, 0·25 and 0·1 and were plotted in Fig. 16. The plotting is made easier if the location of the maximum of the curve is known. Naturally (P/P_{cr}) has its maximum value at the same value of a as Π/Π_{cr}. The slope of the Π/Π_{cr} curve is

$$\frac{\partial (\Pi/\Pi_{cr})}{\partial a} = \frac{(1/a^2) - (1/2)(a/e) + (3/4)}{[1 + (a/2)^2]^2} e. \quad . \quad (202)$$

The slope is zero when

$$e = (a^3/2)/[1 + (3/4) a^3]. \quad . \quad . \quad (203)$$

DYNAMICS OF THE MOTION OF THE COLUMN

The motion of the column is governed by the equation

$$\partial^2 M/\partial x^{*2} + P\partial^2 (y_{ad} + y_0)/\partial x^{*2} - \mu A \partial^2 y_{ad}/\partial t^2 = 0 \quad . \quad (204)$$

where P is positive if it is an axial tensile force. This equation can be written in a non-dimensional form if the following definitions are introduced:

$$\begin{aligned} &M = (1/4) AKh^* m && P = (1/2) AKp \\ &x^* = Lx && t = (L\varepsilon_E/c)\xi \\ &y_{ad} = \rho w_{ad} \qquad y_0 = \rho w_0 && w = w_0 + w_{ad} \end{aligned} \quad . \quad . \quad . \quad (205)$$

With these definitions one obtains

$$(\partial^2 w/\partial \xi^2) - \Phi [p (\partial^2 w/\partial x^2) + (\partial^2 m/\partial x^2)] = 0 \quad (206)$$

where the non-dimensional dynamic similarity number Φ is defined as

$$\Phi = K\varepsilon_E^2/2\mu c^2. \quad . \quad . \quad . \quad (207)$$

In view of equations (148) and (151), namely

$$E_t = (K/n)/\varepsilon^{(n-1)/n} \quad \text{and} \quad \varepsilon_{cr} = \varepsilon_E/n$$

equation (207) can also be written in the form

$$\Phi = (E_{t,cr}/\mu c^2)(n^3/2)\, \varepsilon_{cr}^{(3n-1)/n} \quad . \quad (207a)$$

or with $n = 3$, as

$$\Phi = (E_{t,cr}/\mu c^2)(27/2)\, \varepsilon_{cr}^{8/3}. \quad . \quad . \quad (207b)$$

It can be seen that Φ is proportional to the square of the ratio of the sound velocity at the critical stress to the loading velocity and to a power of the critical strain.

On the basis of the results obtained from the static analysis, the deflected shape will be assumed to be a simple sine curve. With a similar initial deviation from straightness one can write

$$\begin{aligned} w_0 &= e \sin \pi x \\ w_{ad} &= g \sin \pi x \\ m &= b \sin \pi x. \end{aligned}$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (208)$$

Substitution in equation (206) yields

$$(\partial^2 g/\partial \xi^2) + \pi^2 \Phi [(g + e) p + b] = 0. \quad . \quad (209)$$

However, in this differential equation p and b are functions of a and ξ. One connection between the variables can be obtained through calculation of the additional curvature of the column. From the expression for the lateral deflections one gets

$$(1/r) = -(\partial^2 y_{ad}/\partial x^{*2}) = +(\pi/L)^2 g\rho \sin \pi x. \quad (210)$$

On the other hand, from equations (158), (164) and (165)

$$(1/r) = (1/2)^n (1/h^*)[(p + m)^n - (p - m)^n]. \quad . \quad (211)$$

When $n = 3$, this equation becomes

$$(1/r) = (1/4h^*)(3p^2 m + m^3). \quad . \quad . \quad (212)$$

As p is proportional to the axial load, it is independent of x. The value of m can be substituted from the last of equations (208); in doing so, one should use the approximation

$$\sin^3 \varphi = (3/4) \sin \varphi$$

because the higher harmonics are disregarded in this analysis. Substitutions yield

$$(1/r) = +(3/16h^*)(4p^2 b + b^3) \sin \pi x. \quad . \quad (213)$$

From a comparison of equations (210) and (213) one obtains

$$4p^2 b + b^3 = (32/3)\, \varepsilon_E g. \quad . \quad . \quad (214)$$

DISPLACEMENT LOADING IN ELASTIC TESTING MACHINE

The column is assumed to be loaded in an elastic testing machine. The elasticity of the machine is taken into account in the manner shown schematically in Fig. 17. The upper end point of the column is guided along a straight line and is connected with a rigid plunger by means of a spring whose constant is C. The plunger is moving down with a velocity c.

At time t the displacement of the plunger is ct. This must be equal to the sum of the shortening ΔL_{spring} of the spring (at $t = 0$ there is no force and no displacement), of the shortening ΔL_{curv} of the distance between the end points of the column in consequence of the curvature, and the shortening ΔL_{comp} of the length of the column because of the compressive strain.

With P positive when tension, the shortening of the spring is

$$\Delta L_{spring} = -P/C = -(AK/2C) p. \quad . \quad (215)$$

The difference between the length of the curved column in its initial shape and the distance between the end points is

$$\Delta L_1 = (1/2) \int_0^L y_0'^2 dx^*$$

where the prime designates differentiation with respect to x^*. In the final deflected shape this difference is

$$\Delta L_2 = (1/2)\int_0^L (y_{ad}' + y_0')^2 dx^*.$$

Hence the displacement of the end point of the column in consequence of the lateral deflections is

$$\Delta L_{curv} = (1/2)\int_0^L [(y_{ad}' + y_0')^2 - y_0'^2]\, dx^*$$

$$= (1/2)\int_0^L (y_{ad}'^2 + 2y_{ad}'y_0')\, dx^*.$$

Substitution and integration yield

$$\Delta L_{curv} = (L/4)\,\varepsilon_E (g^2 + 2ge). \quad . \quad . \quad (216)$$

The average strain in the column can be given in the following form if use is made of equations (164) and (165)

$$\varepsilon_{ave} = (1/2)(\varepsilon_t + \varepsilon_c) = (1/2)^{n+1}[(p+m)^n + (p-m)^n].$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (217)$$

When n is 3, this equation becomes

$$\varepsilon_{ave} = (1/8)(p^3 + 3pm^2). \quad . \quad . \quad (218)$$

Substitution from the last of equations (208) and consideration of the approximation

$$\sin^2\varphi \simeq (1/2)$$

lead to

$$\varepsilon_{ave} = (1/16)[2p^3 + 3pb^2].$$

Hence the shortening of the column due to uniform compression is

$$\Delta L_{comp} = -(L/16)(2p^3 + 3pb^2). \quad . \quad (219)$$

The total shortening of the column is then from equations (215), (216) and (219)

$$\Delta L_{tot} = -(AK/2C)\,p + (L/4)\,\varepsilon_E (g^2 + 2ge) -$$
$$-(L/16)(2p^3 + 3pb^2).$$

This must be equal to the plunger displacement ct at all times. The condition can be written in the form

$$(ct/L\varepsilon_E) = \dot{\xi} = (1/4)(g^2 + 2ge) -$$
$$-Sp - (1/16\varepsilon_E)(2p^3 + 3pb^2)$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (220)$$

where

$$S = (AK/2C)(1/L\varepsilon_E). \quad . \quad . \quad (220a)$$

The problem of the motion of the column in the elastic testing machine is fully defined by the differential equation (209) and the two conditions presented in equations (214) and (220).

STATIC SOLUTION

These equations will first be solved for static conditions, that is for the case when the inertia forces can be neglected. The equations are collected once more below:

$$(\partial^2 g/\partial\xi^2) + \pi^2\Phi\,[(g+e)\,p + b] = 0 \quad . \quad (209)$$

$$4p^2 b + b^3 = (32/3)\,\varepsilon_E g \quad . \quad . \quad (214)$$

$$\dot{\xi} = (1/4)(g^2 + 2ge) - Sp - (1/16\varepsilon_E)(2p^3 + 3pb^2)$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (220)$$

with $$S = (AK/2C)(1/L\varepsilon_E). \quad . \quad . \quad (220a)$$

The deflections and the moment are assumed as

$$w_{ad} = g \sin \pi x$$
$$w_0 = e \sin \pi x$$
$$w = a \sin \pi x$$
$$m = b \sin \pi x$$
$$w = w_{ad} + w_0 \qquad a = g + e$$
$$y_{ad} = \rho w_{ad} \qquad y_0 = \rho w_0 \qquad y = \rho w.$$

When the inertia term is neglected, equation (209) reduces to

$$(g+e)\,p + b = 0.$$

Hence $$b = -ap. \quad . \quad . \quad . \quad (221)$$

Substitution in equation (214) and manipulations yield

$$(\Pi/\Pi_{cr}) = [1 - (e/a)]/[1 + (a/2)^2]$$

which is identical with the solution found earlier and presented as equation (200).

The displacement of the plunger of the testing machine as given in equation (220) becomes upon substitution of $b = -ap$

$$\dot{\xi} = (1/4)(a^2 - e^2) - (1/16\varepsilon_E)(2 + 3a^2)\,p^3 - Sp$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (222)$$

because $$g^2 + 2ge = a^2 - e^2.$$

As from equation (172)

$$p^3 = 16(\rho/L)^2\Pi$$

and from equation (185a)

$$\Pi_{cr} = -\pi^2/6$$

one obtains $$p^3 = -(8/3)\,\varepsilon_E (\Pi/\Pi_{cr}). \quad . \quad . \quad (223)$$

Substitution in equation (222) yields

$$\dot{\xi} = (1/4)(a^2 - e^2) + (1/6)(2 + 3a^2)(\Pi/\Pi_{cr}) - Sp.$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (224)$$

Because of equation (200) this can also be written as

$$\dot{\xi} = (1/4)(a^2 - e^2) + (2/3)[1 - (e/a)]$$
$$[(2 + 3a^2)/(4 + a^2)] - Sp. \quad . \quad (225)$$

In equation (225) the value of p as well as that of S depends upon the slenderness ratio of the column. Hence $\dot{\xi}$ could be computed for an elastic testing machine only after the slenderness ratio was chosen. In all the numerical work here presented $L/\rho = 50$. This gives $\varepsilon_E = 3{\cdot}95 \times 10^{-3}$. From equation (223)

$$p = -2(\varepsilon_E/3)^{1/3}(\Pi/\Pi_{cr})^{1/3} = -2(\varepsilon_E/3)^{1/3}(P/P_{cr}).$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (226)$$

With the value $\varepsilon_E = 3{\cdot}95 \times 10^{-3}$ this becomes

$$p = -0{\cdot}2193\,(P/P_{cr}). \quad . \quad . \quad (227)$$

The results of the computations are presented in Fig. 18.

SLOW LOADING IN TESTING MACHINE

Equations (209), (214) and (220) can be written as

$$(\partial^2 a/\partial\xi^2) + \pi^2\Phi\,(ap + b) = 0 \quad . \quad . \quad (228)$$

$$4p^2 b + b^3 = (32/3)\,\varepsilon_E (a - e) \quad . \quad . \quad (229)$$

$$\dot{\xi} = (1/4)(a^2 - e^2) - Sp - (1/16\varepsilon_E)(2p^3 + 3pb^2)$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (230)$$

with $$S = (AK/2C)(1/L\varepsilon_E). \quad . \quad . \quad (231)$$

When the loading takes place very slowly, the motion of the column can be expected to be essentially the motion corresponding to the static equilibrium positions. It will be assumed, therefore, that at every value of ξ the solution can be expressed in the form

$$a = a_{st} + \alpha \quad . \quad . \quad . \quad (232a)$$

$$b = b_{st} + \beta \quad . \quad . \quad . \quad (232b)$$

$$p = p_{st} + \varphi \quad . \quad . \quad . \quad (232c)$$

where the subscript "st" designates the static solution and the perturbations represented by the Greek letters, all functions of ξ, are small as compared to the static solutions. If these expressions are substituted into equations (228) to (231), squares and products of the perturbations are disregarded, and finally the subscript "st" is omitted for the sake of simplicity, the following equations are obtained:

$$(\partial^2\alpha/\partial\xi^2) + \pi^2\Phi(a\varphi + p\alpha + \beta) = -(\partial^2 a/\partial\xi^2) \quad (233)$$

$$4p^2\beta + 8pb\varphi + 3b^2\beta = (32/3)\varepsilon_E\alpha \quad . \quad (234a)$$

$$(1/2)a\alpha - S\varphi - (1/16\varepsilon_E)(6p^2\varphi + 6bp\beta + 3b^2\varphi) = 0.$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (234b)$$

In these equations a, b, and p represent the static solution. A proof resembling that given in Appendix II would show that in most cases $(\partial^2 a/\partial\xi^2)$ is negligibly small; consequently equation (233) can be replaced by

$$(\partial^2\alpha/\partial\xi^2) + \pi^2\Phi(a\varphi + p\alpha + \beta) = 0. \quad . \quad (234c)$$

Now φ and β will be expressed in terms of α. First it is recalled that in the static solution $b = -ap$. With this substitution equation (234a) becomes

$$(4 + 3a^2)\beta - 8a\varphi = (32/3)(\varepsilon_E/p^2)\alpha. \quad . \quad (235)$$

Similarly equation (234b) can be written in the form

$$(8/3)(\varepsilon_E/p^2)a\alpha = (16/3)(\varepsilon_E/p^2)S\varphi + (2 + a^2)\varphi - 2a\beta.$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (236)$$

After elimination of the terms containing α one obtains

$$(\beta/\varphi) = (4/3a)[(16/3)(\varepsilon_E/p^2)S + 2 + 3a^2]/(4 + a^2). \quad (237)$$

Substitution of β from equation (237) into equation (235) and solution for φ yields

$$(a\varphi/\alpha) = \frac{8a^2(4 + a^2)(\varepsilon_E/p^2)}{(16/3)(4 + 3a^2)(\varepsilon_E/p^2)S + 8 - 6a^2 + 3a^4}. \quad (238)$$

Finally from equation (237) one obtains

$$(\beta/\alpha) = (32/3)(\varepsilon_E/p^2) \times$$

$$\times \frac{(16/3)(\varepsilon_E/p^2)S + 2 + 3a^2}{(16/3)(4 + 3a^2)(\varepsilon_E/p^2)S + 8 - 6a^2 + 3a^4}. \quad (239)$$

Equation (234c) can now be written as

$$(\partial^2\alpha/\partial\xi^2) + h^2\chi\alpha = 0 \quad . \quad . \quad (240)$$

where $$\chi = (8\pi^2/3)(\varepsilon_E/p^2)(N/D) \quad . \quad . \quad (241a)$$

and

$$N = [3a^4 + 24a^2 + 8 + (64/3)(\varepsilon_E/p^2)S] -$$

$$-(\Pi/\Pi_{cr})[(16/3)(\varepsilon_E/p^2)(4 + 3a^2)S + 8 - 6a^2 + 3a^4]$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (241b)$$

$$D = (16/3)(\varepsilon_E/p^2)(4 + 3a^2)S + 8 - 6a^2 + 3a^4$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (241c)$$

$$h^2 = \Phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (241d)$$

Appendix IV

Rapid Loading of Non-linear Column

When the non-linear column is loaded rapidly in the testing machine, the dynamic similarity number Φ is small. Under such conditions the deflection increases with time without oscillations, at least up to the instant when the load has its maximum value. The solution of equations (209), (214) and (220) can then be obtained by expansion in power series.

If the notation is introduced

$$b^* = 2\varepsilon_E{}^{1/3}b$$

$$p^* = \varepsilon_E{}^{1/3}p$$

$$g^* = 2g$$

$$\xi^* = \xi/[\pi^2\Phi\varepsilon_E{}^{1/3}]^{1/2}$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (242)$$

where the starred quantities are the same as the ones used in Appendix III without the stars, and the constants are assumed to have the following values

$$\pi^2\Phi = 10 \qquad L/\rho = 50 \qquad S = 10 \quad . \quad (243)$$

we obtain

$$\varepsilon_E{}^{1/3} = 0{\cdot}15805 \qquad (\pi^2\Phi\varepsilon_E{}^{1/3})^{1/2} = 1{\cdot}2572 \quad (244)$$

and the basic equations become

$$(d^2g/d\xi^2) + [g + (e/2)]p + b = 0 \quad . \quad (245)$$

$$p^2b + b^3 = (8/3)g \quad . \quad . \quad . \quad (246)$$

$$\xi = 1{\cdot}2572(g^2 + ge) - 1{\cdot}987p - 0{\cdot}1572(p^3 + 6pb^2)$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (247)$$

The initial conditions are

$$g(0) = 0 \qquad dg/d\xi = 0 \quad \text{when} \quad \xi = 0$$

$$b(0) = 0 \qquad p(0) = 0 \quad . \quad (248)$$

It follows from equations (209) and (248) that the power series representing g in terms of ξ must begin with the ξ^3 term and the series representing b and p with the ξ term. We may write therefore

$$g = g_3\xi^3 + g_4\xi^4 + g_5\xi^5 + g_6\xi^6 + g_7\xi^7 + g_8\xi^8 + \ldots \quad . \quad (249)$$

$$b = b_1\xi + b_2\xi^2 + b_3\xi^3 + b_4\xi^4 + b_5\xi^5 + b_6\xi^6 + \ldots \quad . \quad (250)$$

$$p = p_1\xi + p_2\xi^2 + p_3\xi^3 + p_4\xi^4 + p_5\xi^5 + p_6\xi^6 + \quad . \quad (251)$$

Substitution in equation (245) yields, with $e = 0{\cdot}01$,

$$0{\cdot}005p_1 + b_1 + 6g_3 = 0 \quad . \quad . \quad (252a)$$

$$0{\cdot}005p_2 + b_2 + 12g_4 = 0 \quad . \quad . \quad (252b)$$

$$0{\cdot}005p_3 + b_3 + 20g_5 = 0 \quad . \quad . \quad (252c)$$

$$. \quad . \quad .$$

$$. \quad . \quad .$$

$$. \quad . \quad .$$

Substitution in equation (246) gives

$$p_1{}^2b_1 + b_1{}^3 - (8/3)g_3 = 0 \quad . \quad . \quad (253a)$$

$$2p_1p_2b_1 + p_1{}^2b_2 + 3b_1{}^2b_2 - (8/3)g_4 = 0 \quad . \quad (253b)$$

$$p_2{}^2b_1 + 2p_1p_3b_1 + 2p_1p_2b_2 +$$

$$+ p_1{}^2b_3 + 3b_2{}^2b_1 + 3b_1{}^2b_3 - (8/3)g_5 = 0 \quad (253c)$$

$$. \quad . \quad .$$

$$. \quad . \quad .$$

$$. \quad . \quad .$$

Similarly one obtains from equation (247)

$$-1{\cdot}987\,p_1-1=0 \quad . \quad . \quad . \quad (254a)$$

$$-1{\cdot}987\,p_2=0 \quad . \quad . \quad . \quad (254b)$$

$$0{\cdot}01257\,g_3-1{\cdot}987\,p_3-0{\cdot}1571\,p_1{}^3-0{\cdot}9429\,p_1b_1{}^2=0 \quad . \quad . \quad . \quad . \quad (254c)$$

. . .
. . .
. . .

The first set of three equations (equations (252*a*), (253*a*) and (254*a*)) contains the three unknowns p_1, b_1, and g_3. Of these, p_1 is determined by equation (254*a*). The two remaining equations contain therefore only two unknowns. If g_3 is eliminated between the two equations, the result is a cubic for b_1. The cubic has one real root which is positive.

The second set of equations (equations (252*b*), (253*b*) and (254*b*)) contains the unknowns p_2, g_4 and b_2. Of these, p_2 is determined by equation (254*b*) alone. The remaining two equations are linear in b_2 and g_4. Consequently the values of the coefficients can be computed without difficulty. Substitution of the coefficients and of a value $\xi=\xi_1$ in equations (249) to (251) yields the values of g, b, and p at ξ_1. However, the value of ξ_1 must be chosen small if the coefficients contained in the three sets of equations are to describe the state of the column with sufficient accuracy. More terms are not convenient to use because the numerical work involved increases rapidly with the number of terms taken although no difficulties in principle arise.

When the numerical solution valid at $\xi=\xi_1$ is obtained, a new expansion about this point is required. In the new series the first few terms g_0, g_1, ... are present because at $\xi=\xi_1$ the quantities g, $dg/d\xi$, p, and b do not vanish. One may write

$$g=g_0+g_1j+g_2j^2+g_3j^3+g_4j^4+\ldots \quad . \quad (255)$$

$$p=p_0+p_1j+p_2j^2+\ldots \quad . \quad . \quad . \quad (256)$$

$$b=b_0+b_1j+b_2j^2+\ldots \quad . \quad . \quad . \quad (257)$$

where $g_0=g(\xi_1)$, $g_1=g'(\xi_1)$, $p_0=p(\xi_1)$, $b_0=b(\xi_1)$ are known.

The basic equations are now

$$(d^2g/dj^2)+[g+(e/2)]\,p+b=0 \quad . \quad (258)$$

$$p^2b+b^3=(8/3)\,g \quad . \quad . \quad . \quad (259)$$

$$j=-\xi_0+1{\cdot}257\,(g^2+ge)-1{\cdot}987\,p-1{\cdot}571\times10^{-1}\,(p^3+6\,pb^2) \quad . \quad . \quad . \quad . \quad (260)$$

where ξ_0 is a known constant.

Substitution of the expressions given in equations (255) to (257) yields the conditions

$$2g_2+[g_0+(e/2)]\,p_0+b_0=0 \quad . \quad (261a)$$

$$6g_3+[g_0+(e/2)]\,p_1+g_1p_0+b_1=0 \quad . \quad (261b)$$

$$12g_4+[g_0+(e/2)]\,p_2+g_1p_1+g_2p_0+b_2=0 \quad . \quad (261c)$$

. . . .
. . . .
. . . .

$$p_0{}^2b_0+b_0{}^3-(8/3)\,g_0=0 \quad . \quad (262a)$$

$$p_0{}^2b_1+2p_0p_1b_0+3b_0{}^2b_1-(8/3)\,g_1=0 \quad . \quad (262b)$$

$$(2p_0p_2+p_1{}^2)\,b_0+2p_0p_1b_1+p_0{}^2b_2+3b_0{}^2b_2+3b_0b_1{}^2-(8/3)\,g_2=0 \quad . \quad (262c)$$

. . .
. . .
. . .

$$-\xi_0+1{\cdot}257\,(g_0{}^2+g_0e)-1{\cdot}987p_0-1{\cdot}571\times10^{-1}\,(p_0{}^3+6p_0b_0{}^2)=0 \quad . \quad . \quad . \quad . \quad (263a)$$

$$1{\cdot}257\,(2g_0g_1+g_1e)-1{\cdot}987p_1-1{\cdot}571\times10^{-1}\,[3p_0{}^2p_1-12b_0b_1p_0+6p_1b_0{}^2]-1=0 \quad . \quad . \quad . \quad . \quad (263b)$$

$$1{\cdot}257\,[2g_0g_2+g_1{}^2+g_2e]-1{\cdot}987p_2-1{\cdot}571\times10^{-1}\,[3p_0{}^2p_2+3p_0p_1{}^2+6p_0\,(2b_0b_2+b_1{}^2)+12b_0b_1p_1+6p_2b_0{}^2]=0 \quad . \quad . \quad (263c)$$

When the values of the coefficients are computed and substituted in equations (255) to (257), together with $j=j_1$, the state of the column is known at $\xi=\xi_1+j_1$. In order to continue with the calculations a new expansion about $j=j_1$ is needed. This is of the same form as equations (255) to (257). Thus the process can be continued until the maximum value of the load is reached.

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100. TYLER, C. M., JR. (1951). Dynamic Buckling of Columns. *Journal of Applied Mechanics,* Vol. 18, No. 3, p. 317. September 1951.
The dynamics of the buckling process of thin rectangular panels is treated in:
101. ZIZICAS, G. A. (1952). Dynamic Buckling of Thin Elastic Plates. *Transactions of the American Society of Mechanical Engineers,* Vol. 74, No. 7, p. 1257, Oct. 1952.

After this paper was completed the following publications, dealing with dynamic buckling under loads increasing linearly with time, have come to the author's attention:

102. KATSUTA, CHITOSHI (1947). On Buckling of an Elastic Long Column under High-Speed Loading. Paper read before a meeting of the Architectural Institute of Japan, 23rd November 1947. Library of Congress reproduction PB 94937.
103. SUHARA, JIRÔ (1948). Dynamic Stability of Column Compressed by Axial Load Increasing with Time. Paper read before the Research Committee on Applied Mechanics, Faculty of Engineering, University of Kyushu, May 1948. Library of Congress reproduction PB 94945.
104. KATSUTA, CHITOSHI (1952). High-Speed Buckling of Mild Steel Columns. *Proceedings of the First Japan National Congress for Applied Mechanics,* Japan National Committee for Theoretical and Applied Mechanics, Science Council of Japan, p. 65, May 1952.

Creep buckling is discussed in:

105. MIYAGAWA, MATUO (1952). On Buckling Rupture of Columns Caused by Creep. *Proceedings of the First Japan National Congress for Applied Mechanics,* Japan National Committee for Theoretical and Applied Mechanics, Science Council of Japan, p. 47, May 1952.

The buckling of a column hit axially by a mass is analysed in:

106. IMACHI, ISAMU. On the Axial Impact on a Thin Strut Considering its Lateral Stability. Library of Congress Reproduction PB 94940.

MR. GEORGE R. EDWARDS, C.B.E., *Fellow* (Vice-President of the Society), proposing a vote of thanks to the lecturer, said that he had sat and marvelled at Professor Hoff, who virtually had not used a note and had spoken for nearly one hour without any hesitation and with absolute mastery. It was an incredible performance.

The subject of struts had been worrying people in the structural side of aviation for many years and he could just remember, what seemed a long time ago, feeling very bitter about the fact that the Euler strut never did seem to be there in practice, and that end fixations always seemed to be the controlling factor. In this country the assumptions that were made on the end fixation condition of a strut made the reasoning of the people who built the Greek temple columns almost scientific!

The general gist of Professor Hoff's message seemed to be that as usual in the aviation business things were not always what they seemed. He had been somewhat depressed to think that there were going to be still bigger differences between the results that the testing machine was alleged to substantiate and what really happened, and he had become thoroughly depressed when, near the end of the paper, Professor Hoff had introduced another reason for the creeps! However, he had brightened considerably when, in his closing sentence, the lecturer had said that the trouble the fellow caused in the shops did not matter too much; as long as one put in a little more area than one thought was needed it would be all right in the end, which was what almost every stressman on either side of the Atlantic always did.

In expressing to Professor Hoff the gratitude of the Society, Mr. Edwards said the Wilbur Wright Memorial Lectures in general took the form of a more general review of the field but on this occasion the lecturer had attacked a specialised subject. It was a wonderful beginning to the Anglo-American Aeronautical Conference.

Following the lecture a Reception was held at the headquarters of the Society, 4 Hamilton Place, London, W.1, which was attended by well over 700 members, guests and delegates to the Fourth Anglo-American Conference.

Foaming and Aeration of Oils in Aviation Powerplant

by

R. TOURRET, B.Sc.(Eng.), A.M.I.Mech.E., A.F.Inst.Pet., Grad.R.Ae.S.

(Thornton Research Centre)

1. Introduction

The presence of foam or unduly aerated oil in the lubricating-oil systems of aircraft power plant can be very troublesome[1]—more so in reciprocating engines than in gas turbines, since the rates of circulation are greater and the viscosity grade of the oil is higher. Consequently in this paper, which surveys present knowledge about oil foaming, reciprocating engines are the main consideration.

1.1. AIRCRAFT-ENGINE OIL SYSTEMS

In the conventional aircraft "dry-sump" oil system, a simplified diagram of which is shown in Fig. 1, oil is pumped from a supply tank to the bearings, and so on, by a so-called "pressure" pump. This oil then drains down into the sump and is picked up by "scavenge" pumps which return it to the supply tank. There are usually at least two scavenge pumps so that the oil is cleared away irrespective of the attitude of the aircraft. A common arrangement is to have one scavenge pump at the front of the engine (which clears the oil when the aircraft is diving) and one at the rear (which deals with climbing conditions). Since each scavenge pump will usually have a greater capacity than the pressure pump to ensure that the oil level does not build up in the sump, the excess capacity will pump crankcase gases into the oil system unless special arrangements are made.

These vapours will be returned with the oil to the supply tank and, unless there separated from the oil, will be picked up by the pressure pump and circulated round the engine. If a "hot-pot" or warm-up compartment is fitted in the oil tank to facilitate cold starting of the engine by circulating only part of the normal oil supply, foaming is more likely to occur because the small quantity of oil in circulation allows less time for the air to separate out in the oil tank. If the oil is diluted with gasoline to facilitate starting by lowering the viscosity of the oil, a procedure often necessary when operating reciprocating engines in cold climates[2], the evaporation of the gasoline as the oil warms up naturally increases the amount of foam formed.

2. Forms in which Gas may be Dispersed in Oil

Gas may be dissolved or entrained in the oil or it may produce foam. The amount of gas that can be dissolved depends on the solubility coefficient of the particular gas, but there is hardly any limit to the amount of gas that can be entrained in the form of bubbles in the oil. Bubbles leaving the bulk of the oil form foam.

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2.1. DISSOLVED GAS

The solubility coefficient is proportional to the partial pressure of the gas and, in comparing solubilities, it is usual to refer them to a standard pressure. If this is taken as 760 mm. Hg., the solubility is known as Bunsen's Coefficient. Although the value of Bunsen's Coefficient varies for oils of different viscosity and composition, for a typical lubricating oil the values are 0·0783 for air, 0·0662 for oxygen, 0·124 for nitrogen and 0·853 for carbon dioxide. The high value for carbon dioxide is significant because of its presence in the crankcase gases of reciprocating engines.

At normal ground-level temperatures and pressures, an oil is capable of holding about 8 per cent. by volume of air in solution. Decrease in pressure and increase of temperature, such as occurs when climbing at the start of a flight, will cause the solubility to decrease and thus, if the oil is saturated initially, vapour will be released which may cause foam. Fig. 2 shows the theoretical effect of altitude on the volume of air released from oil at 180°F[3]. This takes no account of the effect of any water or gasoline dissolved in the oil, which may increase the volume of vapour released considerably.

2.2. ENTRAINED GAS

Entrained gas consists of bubbles of gas in the oil. These bubbles tend to rise according to Stokes' Law for viscous flow* which states that the velocity of rise increases with the size of the bubble and, because of its effects on the viscosity of the oil, also with the temperature of the oil[1, 4], as follows:—

$$V = \frac{2ga^2 (d_2 - d_1)}{9\eta},$$

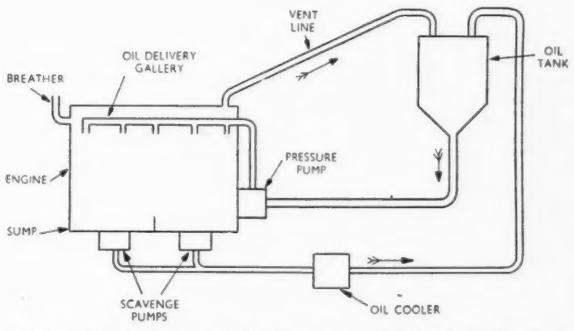

FIGURE 1. Typical aircraft "dry-sump" lubricating oil system.

* Although the rate of oil circulation in the pipes, supply tanks and elsewhere may be high enough for the oil flow to be turbulent, the rate at which the bubbles rise appears to be always low enough for viscous flow conditions to apply.

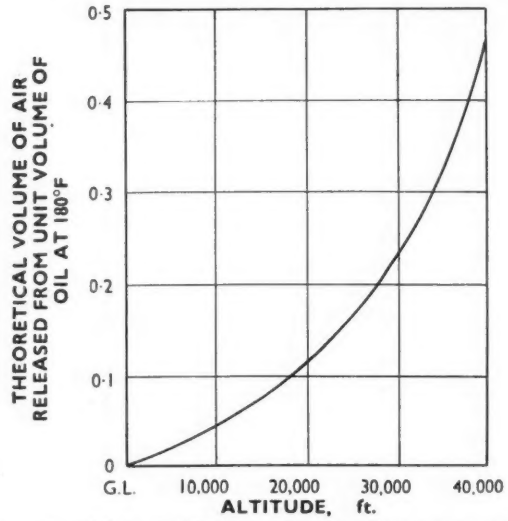

FIGURE 2. Effect of altitude on the theoretical volume of air released from unit volume of oil at 180°F.

where V = velocity of rise of bubble (cm./sec.)
g = acceleration due to gravity, say 980 cm./sec.^{2}
a = radius of bubble, assumed spherical (cm.)
d_1 = density of gas inside the bubble (gm./c.c.)
d_2 = density of oil (gm./c.c.)
η = absolute viscosity of oil (poises).

The density of the gas in the bubble being so much less than that of the oil, d_1 can be neglected with an error of only about 0·1 per cent.

Thus, the smaller the bubbles, the longer they take to reach the surface, where they either break and disappear or form foam. Since small bubbles are less fragile and consequently more likely to form foam than large bubbles, and since anti-foaming methods and agents are generally not effective until the bubbles have left the oil, small bubbles are most troublesome in service. It is thus unfortunate that the passage of vapour through the gear type of pump chiefly used in oil systems tends to produce small bubbles which may not coalesce. The production of such small bubbles has been observed[4] experimentally in a pump to which a transparent front had been fitted.

It had been shown[1] that the rate of rise of bubbles in straight mineral oils obeys Stokes' Law. With certain additive-type oils, however, the rate of rise is less and it is thought that this discrepancy is due to the formation of some quasi-solid structure or layer round the bubble which, rising through the oil with it, increases its resistance to passage through the oil without increasing its displacement lift. It appears that this "shell" acts as if it had a diameter half as great again as that of the bubble of gas itself[5].

Bubbles of gas in aerated oil in pipes tend to rise to the top of the pipe and, if the pipe is long enough and the flow of oil slow enough, these bubbles may coalesce to form a "slug" of air[1]. This process may continue until the gas and oil are flowing separately in the pipe, the gas above the oil. If this occurs, the fluids tend to reach an equilibrium state in which the gas-flow friction losses equal the oil-flow friction losses; the gas travels faster than the oil and the cross-sectional area of the pipe occupied by the gas shrinks proportionately[4].

Separation of gas from the oil in pipes in this manner facilitates de-aeration when the mixture is returned to the oil tank since, even if the delivery point is submerged, large bubbles of gas are formed which separate more readily than the small bubbles which would otherwise be present. On the other hand, separation of gas in the oil-pipes is naturally liable to cause air-locks and should therefore be avoided.

2.3. FOAM OR FROTH

The formation of foam depends on the mechanical bubble-generating capacity of the system in which the oil is used, *i.e.* on the amount of gas which is driven into the oil. Mechanical agitation will cause foam, although more violent agitation may in turn tend to break it[1] (Fig. 3).

A foam will not form in a system in which bubbles of gas are rising to the surface of the liquid, unless the surface film formed from that liquid is capable of possessing a surface tension which varies over its surface in such a manner that it can establish an equilibrium between its own weight and the surface tension[1, 6]—the greater this capacity, the larger the bubbles which may be formed. With absolutely pure liquids it appears to be impossible to form a foam[3] and this is probably because such fluids have no capacity for variation in surface tension. Very slight traces of impurities (perhaps only of the order of 0·01 per cent.), however, may act as surface-active agents and allow a varying surface tension, the film thickness and the concentration of the surface-active agent tending to adjust themselves automatically to provide the exact surface tension required to maintain each part of the film. The greater the number of impurities, *i.e.* the more heterogeneous the surface (Gibbs[1]) layer, the greater the foaming tendency[7, 8]. The value of the average surface tension affects the average size of the bubbles in a foam: the greater the surface tension, the smaller the bubbles.

The foregoing is only a simplified statement of the problem. There are many theories of foam

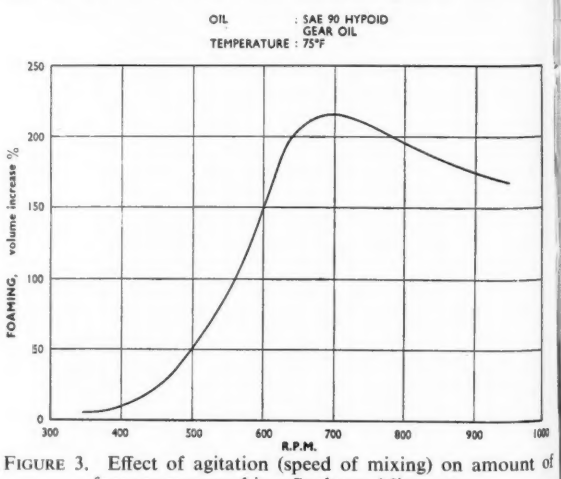

FIGURE 3. Effect of agitation (speed of mixing) on amount of foam as measured in a Sunbeam Mixmaster.

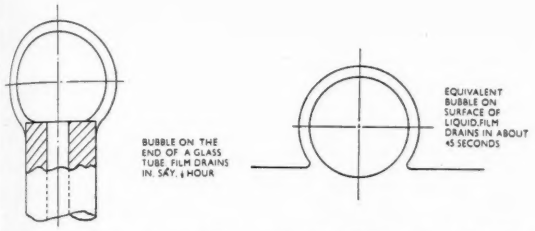
BUBBLE ON THE END OF A GLASS TUBE, FILM DRAINS IN, SAY, ½ HOUR
EQUIVALENT BUBBLE ON SURFACE OF LIQUID; FILM DRAINS IN ABOUT 45 SECONDS

FIGURE 4. Effect of curvature of bubble film on time taken to drain.

formation[9, 10, 11, 12], mostly derived for aqueous solutions, although their applicability to oil foams has been reviewed[3].

It has been shown[3] that if the oil forming the film of a bubble in an oil foam drains hydrodynamically as a true Newtonian fluid, then with the oil viscosity normally encountered the film would thin down to molecular dimensions and break within about one second. It is a matter of common observation that in practice bubbles in oil foam may last considerably longer and hence drainage must be hindered in some way. If a column of foam from an aviation oil is held in a glass container, it will remain for at least several minutes if not for an hour or two. Further, after a while when drainage has been largely completed, interference colours can be seen on the films. That "black" films (of about 120Å thick*) and "grey" films (of from 120Å to about 2,000Å thick) can be observed indicates that these films are at least several, and possibly many, molecules thick. Films from some oils do not even drain to this extent.

Findings of this nature led to the early general concept of a "surface viscosity," due either to the film surface or surface zone possessing some plasticity, or to some structure existing within the liquid. For oil films, it now appears[3] most likely that certain effects of molecular orientation extend inward from the gas/oil interface to an appreciable depth within the oil. There is evidence[13, 14, 15] that orientated layers can extend up to 10,000Å into an oil from a metallic surface and it is thought that a similar phenomenon can occur from a gas/oil interface surface.

If a contaminant or foam-promoting additive is present in the oil, a comparatively thick "plastic" film may be formed on the surface, the effects of which may be additional to those of orientation. It is interesting to recall at this stage the fact (Section 2.2) that bubbles in some additive-type oils rise at a slower rate than that predicted by theory, a discrepancy for which it is difficult to account except by postulating the existence of some layer on the surface.

2.3.1. *Water and gasoline contamination*

If water or gasoline is present in the oil, more foaming may be expected, owing to the larger volumes of gases likely to be given off in vaporisation. A troublesome foam composed of small bubbles can be caused by less than 0·1 per cent. water, especially if the aircraft altitude becomes high enough for the boiling point of the water to fall to near the temperature of the oil.

2.3.2. *Chemical contamination*

Certain substances can act as surface-active agents and cause heavy foaming. These may be of three types:—

(*a*) substances which tend to concentrate at the air/oil interface and lower surface tension
(*b*) substances which raise the surface tension
(*c*) substances which form thick insoluble surface films.

Other properties required of the first type of foam promoter, which is the most frequent type, are that it should be soluble or colloidally dispersible and that the surface film, containing a higher concentration of foam promoter than the bulk oil, should possess a surface tension which can vary to suit circumstances.

In practice these chemicals may be inadvertently introduced into the oil system by gasket materials, sealing compounds, and so on. Since less than 0·01 per cent. of some compounds can cause heavy foaming the slightest contamination may cause trouble.

2.3.3. *Oxidation*

Oxidation of the oil has been reported[6] as causing increased foaming, but it is more likely that this is due to longer periods in service, allowing increased contamination with traces of surface-active agents.

3. Foam Decay

The film of a freshly-formed bubble consists of oil, except for an adsorbed surface film at each gas/oil interface[6]. In the initial "wet" stage, the oil drains from the foam under gravity. The volume of oil drained depends mainly on the initial volume of foam and on the density and viscosity of the oil, obeying the normal hydrodynamic laws. Later, when the main bulk of the oil has left the foam, the effect of surface tension in drawing oil into the interstices between the bubbles becomes important. The greater the curvature at the bubble interstices the greater the surface tension and the more rapidly the oil leaves the bubble film—for

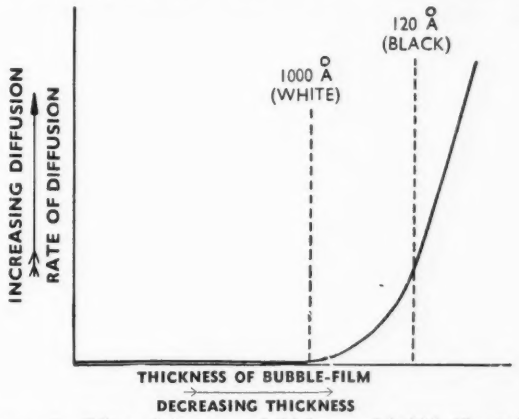

FIGURE 5. Effect of decrease of thickness of bubble-film (with time) on rate of diffusion.

* Å is an Ångstrom unit, equal to 10^{-8} cm.

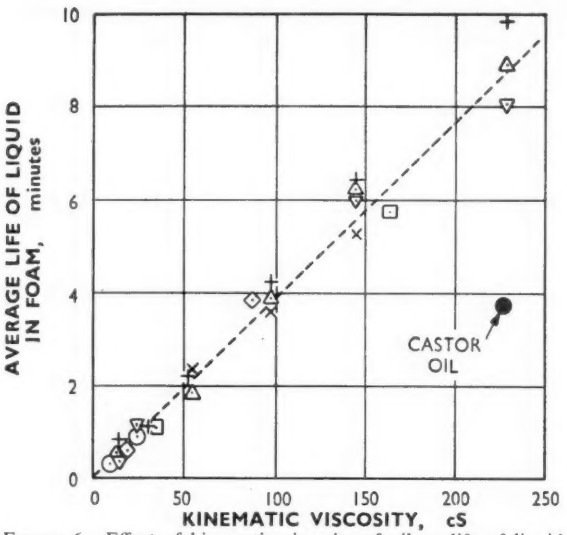

FIGURE 6. Effect of kinematic viscosity of oil on life of liquid in foam for a variety of aviation oils.

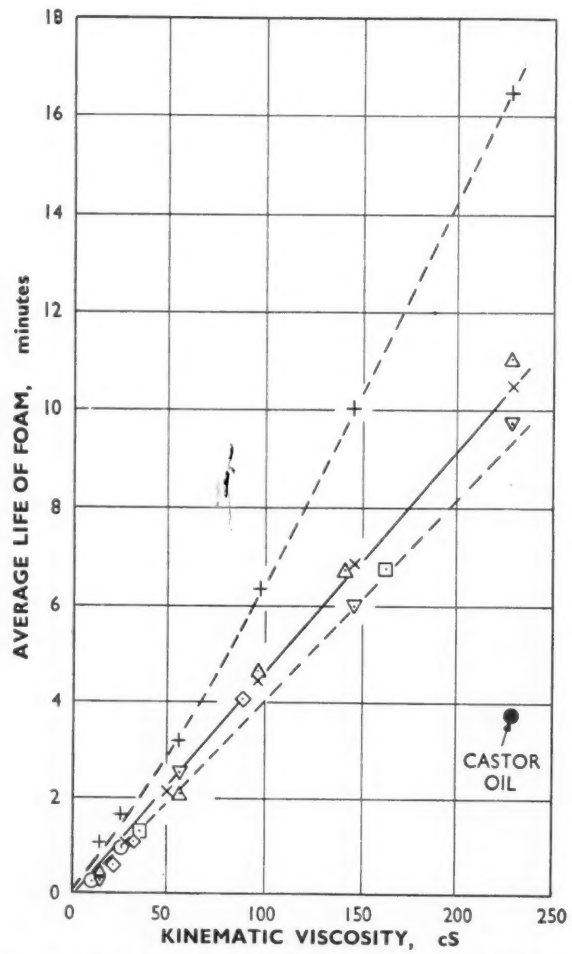

FIGURE 7. Effect of kinematic viscosity of oil on foam life for the same variety of aviation oils as shown in Fig. 6.

example a bubble on the surface of a liquid may drain in of a tube (Fig. 4) owing to the greater curvature at the extremities of the film.

Bubbles on top of the foam, which are exposed to evaporation loss as well as drainage loss, may break if the thickness of the film falls below a certain value[17]. Evaporation losses are especially critical if the component or compounds causing the variable surface tension are more volatile than the oil.

Oil continues to leave the film and the film thickness decreases until interference colours appear. When the thickness of the bubble film decreases to about 1,000Å-2,000Å, being then white in colour, diffusion of gas through the bubble film begins (Fig. 5). After the film thickness reaches about 120Å, corresponding to black, the diffusion rate increases in proportion to the decrease in thickness. Since the pressure within a bubble is one-twentieth of the time taken for a bubble on the end inversely proportional to its diameter*, the smaller bubbles with the larger internal pressures will tend to lose gas and eventually disappear[18]. Similarly, bubbles with convex walls will lose gas to bubbles with plain or concave walls[19]. For a given foam, there is a critical size of bubble—any smaller than this size will diminish and eventually disappear. Liquid drainage in this stage is very small and does not follow a viscosity relationship—the source of the liquid is the interstices between the bubbles, the number of which is reduced as the smaller bubbles disappear. The final product is an extremely fragile "dry" foam which breaks at the slightest mechanical or thermal shock. If it is left completely undisturbed its life may be very long.

Oil foams do not generally pass into the later stages of decay, since the bubbles usually collapse as soon as they have drained sufficiently. The average life of bubbles in an oil foam is therefore proportional to the initial volume of foam and to the viscosity[3, 20] (Figs. 6 and 7). It can be seen that the life of the liquid in the foam (Fig. 6) appears to depend only on viscosity and not on the type of oil, as would be expected from foams formed from undoped oils decaying solely by drainage and not reaching the later stages of foam decay. However, while the life of the foam (Fig. 7) is also in general proportional to the viscosity, the foam life varies for different oils and this is said[3] to be because of differences in the strength of the bubble film.

The temperature of an oil is important because of its influence on viscosity; increased temperature causes a shorter bubble life[21] (Fig. 8) and a more rapid breakdown of foam[3] (Fig. 9). It appears[22] that, in service, if the temperature of the normal viscosity grade oils is above 70°C., foam generally decays sufficiently rapidly so as not to cause excessive build-up (as might be expected from the results shown in Fig. 9).

Since foaming in service is a dynamic process in which foam formation and decay generally occur simultaneously, the amount of foam tends to reach an equilibrium value at which rate of formation is balanced by rate of decay[1]. High-viscosity oils, which make the most persistent foams, offer more resistance to agitation

* $P = 4S/R$, where P is the pressure, S the surface tension and R the radius.

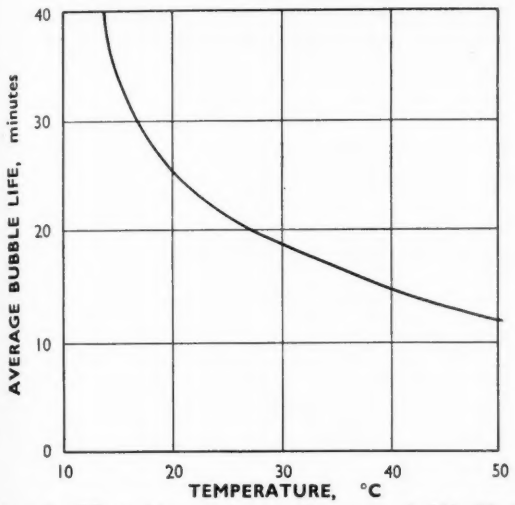

FIGURE 8. Effect of temperature on the average bubble life for a higher hydrocarbon.

and aeration and are consequently less prone to foam formation than low-viscosity oils. Briefly therefore, if an oil has a high viscosity, foam will not form easily, while if the viscosity is low, the foam will drain rapidly and the life of the foam will be short. Results obtained[23] with various lubricants in various foam-promoting devices (Figs. 10 and 11) illustrate this point; there tends to be a critical viscosity for the greatest amount of foam. Since the viscosity depends on the temperature, there is also a critical operating temperature, for any given oil, which gives most foam.

4. Oil-System Performance

The rate of delivery from the gear type of pump is reduced if the oil becomes aerated (Fig. 12). This may aggravate aeration and foaming if, as the pressure pump delivery falls off, more scavenge capacity is available for pumping crankcase gases into the oil system. Since there may be a pressure drop between the pump inlet and the space between the gear teeth due to various losses[24], the pressure at the inlet to the pump should always be appreciably greater than the vaporisation pressure. If the inlet pressure drops too low, the oil delivery is reduced[4] (Fig. 13), as would be expected[25, 26]. Consequently, no unnecessary restrictions should be allowed to remain at the pump inlet. Even a filter at the inlet may severely reduce the output of the pump at high altitudes[27] (Fig. 14). There should be no restrictions in the supply pipe to the pump, which should be as short as possible, of as large a diameter as possible and with bends as few and of as great a radius as possible. Pressurising the oil tank naturally assists the pressure pump to deliver the desired flow. A booster pump, in or immediately adjacent to, the oil tank would help, albeit at some expense of weight and complication.

It is possible[25] to predict the delivery of gear pumps under different operating conditions with fair accuracy. From such considerations, it was shown for a typical pump that the optimum speed for maximum delivery varied for different altitudes (Fig. 15).

It is necessary to know how high an aircraft can fly without suffering from lack of oil. This "lubricating ceiling" cannot be defined in practice, but if it is arbitrarily defined as the point at which the delivery of the pressure pump drops below, say, half the ground-level delivery, it is possible to predict the effect of various changes on the performance of a typical pump[25] (Fig. 16). An increase in the amount of air entrained in the oil causes the "lubrication ceiling" to drop rapidly, as would be expected, each one per cent. volume of entrained air causing the ceiling to drop by over 1,000 ft. Dissolved air in the oil has surprisingly little effect on the "lubrication ceiling," although this effect would be greater if the pipe resistance were large. The actual effect of dissolved air tends to be even less, however, since there is usually an appreciable time lag before the air comes out of solution, when conditions change. Increase in pipe length lowers the "lubrication ceiling" in quite a marked manner, a 10 ft. increase in length lowering the ceiling by about 5,000 ft. Pipe diameter was found to have a considerable effect on the ceiling if the pipes were small, say less than one in. bore, but not otherwise. The critical diameter depends on the capacity of the pump, among other variables.

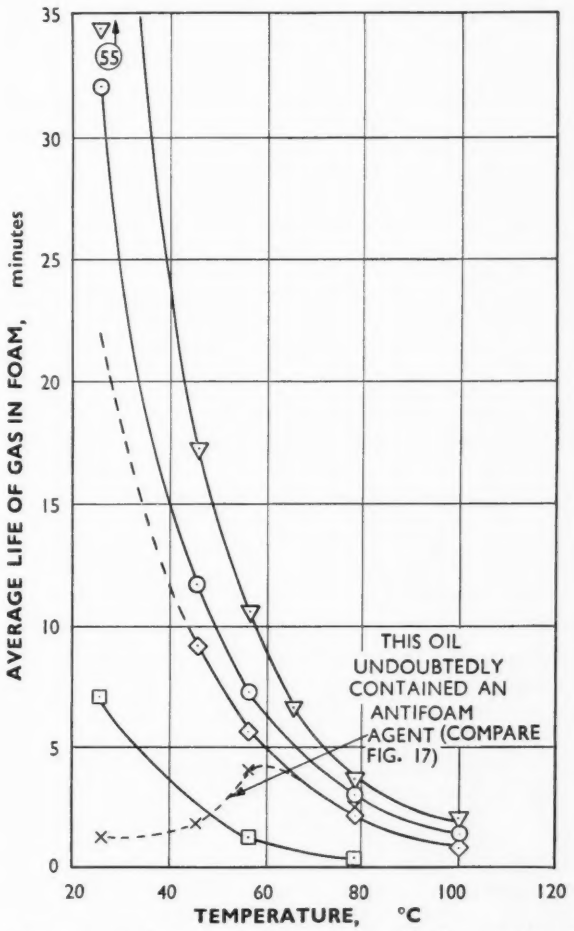

FIGURE 9. Effect of temperature on foam life for a variety of aviation oils.

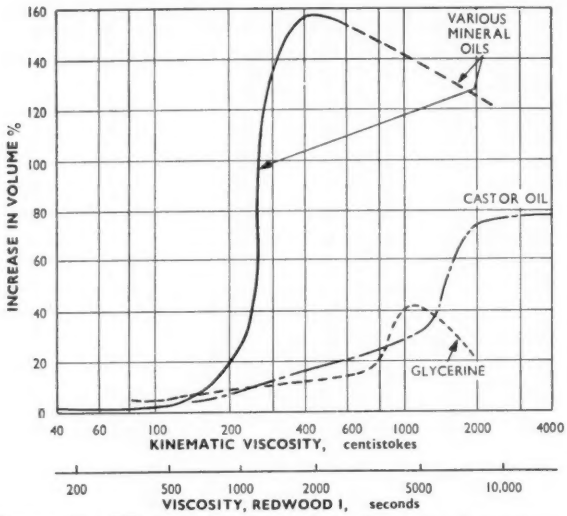

FIGURE 10. Effect of viscosity on foam formation for various fluids.

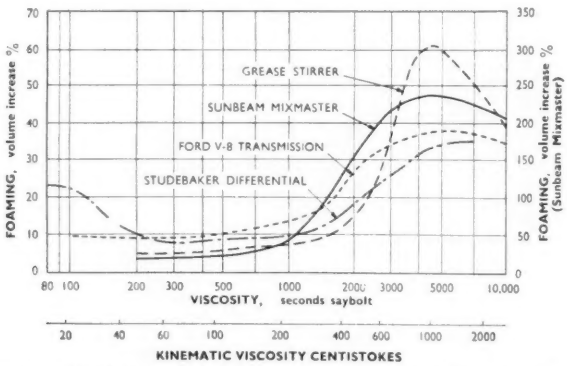

FIGURE 11. Effect of viscosity on amount of foam for gear oils in various foam-promoting devices.

If the position of the supply tank is lowered relative to the pressure pump, the "lubrication ceiling" was found to be lowered. Conversely, if the tank is pressurised, the ceiling is raised by about 9,000 ft. for a tank pressure of one lb./in.^{2}, under the particular conditions studied. The effect of viscosity of the oil on the ceiling or, for the usual viscosity grade of oil, the effect of oil temperature, was found to be that, once above a certain critical temperature or below a certain critical viscosity, there is little change in the height of the "lubrication ceiling." Increase in pump speed was found to cause a considerable drop in the ceiling.

5. Prevention of Foam in Service

The oil supply tank should be designed to facilitate separation of the gas from the aerated oil and thus prevent the pressure pump picking up the gas and circulating it through the oil system. Separation will be facilitated by using a large tank so that the gas has time to separate out, although space and weight are at a premium; and the tank should preferably be shaped so that the oil has a large surface area and a small depth, so as to allow bubbles to reach the surface readily. The oil should be returned to the tank through a submerged pipe (unless impracticable), so that further vapour is not entrained by the jet of oil striking the surface of the oil. The flow path of the oil in the tank should be such as to avoid mixing de-aerated oil with freshly returned aerated oil—at the same time, a strong vortex may facilitate separation of the gas.

Unfortunately, it is difficult to design a "hot-pot" or warm-up compartment, in which only a part of the normal oil supply is circulated so as to facilitate rapid warm-up, without violating some of these principles; hence the take-off and initial climb are often critical periods as regards aeration and foam formation.

Gas may also be removed from the system by special separators, although such devices should only be used as a last resort. One method is to obtain a low-velocity streamline flow which allows time for the bubbles to rise and separate out. Another method is to generate a high radial acceleration, either by a special design of piping or by applying an external force; this separates the gas by centrifugal action.

Foam can be broken by the application of a sharp temperature gradient, mechanical "shock" or a suitable electric field[1, 28], but these methods seem to be unnecessarily complicated for service use.

Probably the best way of minimising aeration and foam in service is to stop putting gas into the oil, *i.e.* to exclude air and crankcase gases from the oil system. Vapour may still appear from any air, water or gasoline dissolved in the oil, but the amount will be far less than the amount of crankcase gases which can be pumped into the oil system.

Aeration by crankcase gases can be prevented by maintaining the scavenge pump pick-ups in a submerged state; a float-level valve can be fitted on the scavenge pumps so that they by-pass a part of the oil back into the sump if the sump level drops so low that there is a danger of the scavenge pump pick-up becoming uncovered. This system, known as the "constant-level" oil sump, appears to be the simplest and most effective method of preventing the vapour from being picked up to form foam, and it is now in common use.

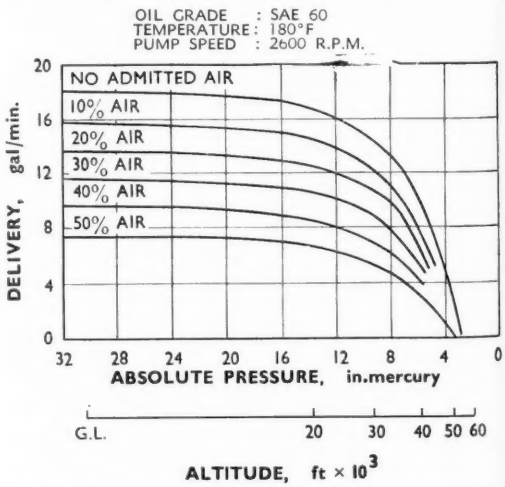

FIGURE 12. Effect of absolute pressure on oil delivery for various amounts of air admitted into the oil.

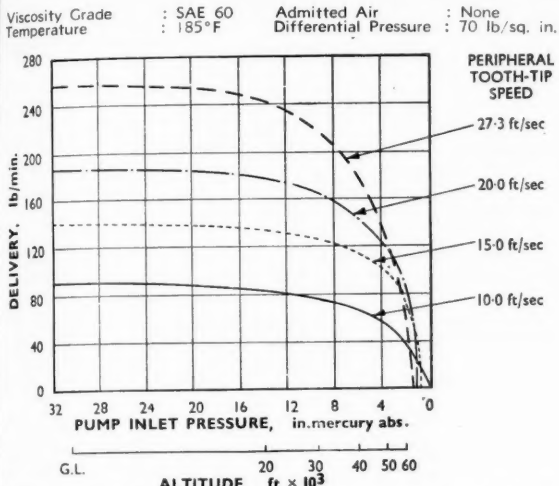

FIGURE 13. Effect of pump inlet pressure on oil delivery for various pump speeds.

These anti-foaming measures should be built into an engine at the design stage. If an existing system is found to be prone to foaming troubles, it may be difficult to make suitable modifications; in this case the trouble may be overcome by the use of a suitable foam inhibitor (marketed oils naturally meet the existing specification). Fig. 17 shows, by way of example, the anti-foaming effect of sorbitol dilaurate and Alronol* when added to a conventional lubricating oil of approximately S.A.E.50 viscosity[3].

However, anti-foaming agents generally cannot greatly affect bubbles while they are in the oil, *i.e.* they can break foams but they cannot de-aerate the oil. A few additives only will affect aerated oil by causing bubbles to coalesce and form larger bubbles which more readily rise to the surface, while others may retard the release of entrained bubbles. (In view of the effect of silicone additives possibly retarding the release of entrained gas from the bulk of the oil, an effect which has been confirmed by practical experiments in aero-engines with non-additive oils, the use of silicones for foam suppression should not be permitted in non-additive aviation oils.) However, since bubbles which are actually in the oil cannot collapse, no matter how unstable they become, anti-foaming agents are far more effective when operating on bubbles in the foam.

The action of an anti-foaming agent is quite complex and several mechanisms may be called into play[2, 29, 30]. Briefly, in the anti-foaming mechanism believed to be most frequently encountered, the agent is attracted by surface-tension forces into the liquid film separating the bubbles. The anti-foaming agent then displaces the foaming agent, the film becomes unstable, and the bubble breaks. Various conditions regarding the ability of an agent first to enter, then to spread within, and finally to rupture the film, have to be satisfied[29, 30] for the particular combination of agent and liquid, before that agent will defoam that particular liquid. This is one reason why anti-foaming agents are not, generally, universally applicable to any liquid.

* Propylated amines.

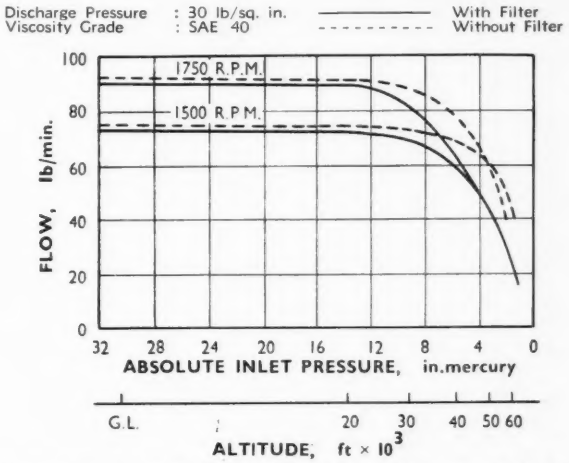

FIGURE 14. Effect of filter at pump inlet on delivery at altitude.

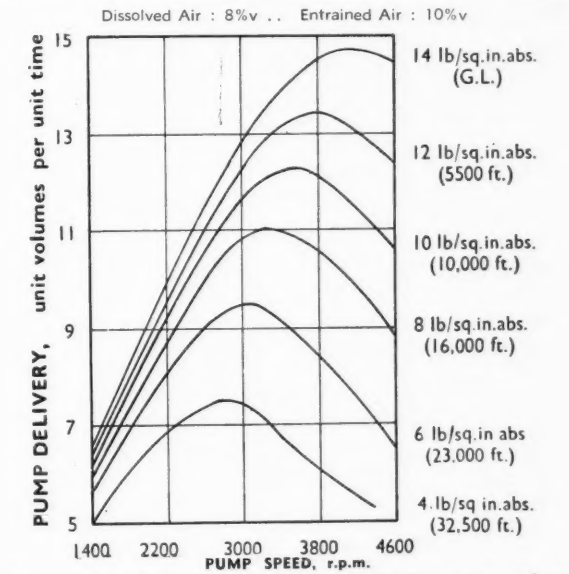
Dissolved Air : 8%v .. Entrained Air : 10%v
PUMP DELIVERY, unit volumes per unit time
14 lb/sq.in.abs. (G.L.)
12 lb/sq.in.abs. (5500 ft.)
10 lb/sq.in.abs. (10,000 ft.)
8 lb/sq.in.abs. (16,000 ft.)
6 lb/sq.in abs (23,000 ft.)
4 lb/sq in.abs. (32,500 ft.)
PUMP SPEED, r.p.m.

FIGURE 15. Effect of pump speed on oil delivery for various altitudes

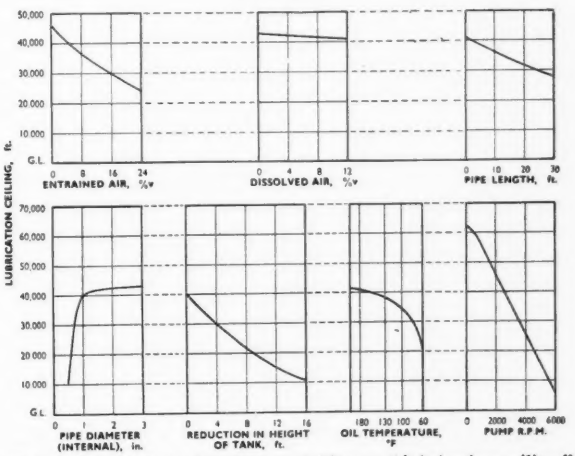

FIGURE 16. Effect of various variables on "lubrication ceiling" of hypothetical oil system.

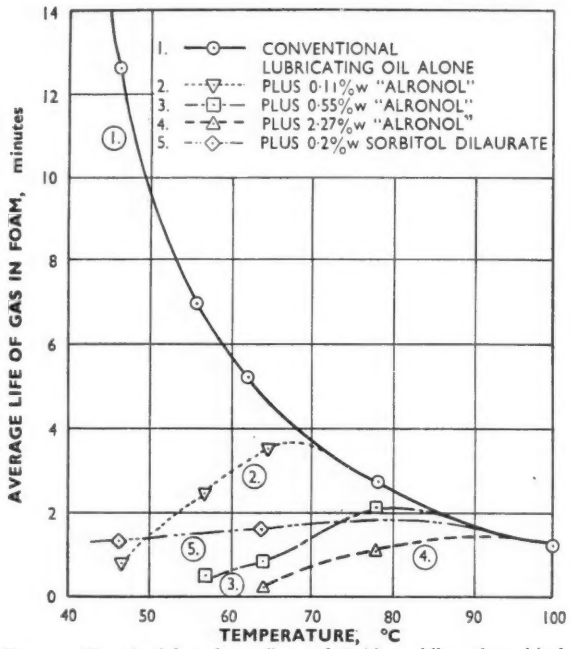

FIGURE 17. Anti-foaming effect of "Alronol" and sorbitol dilaurate on conventional lubricating oil.

Other mechanisms of anti-foaming sometimes encountered include those of a chemical nature[31], for instance a reaction occurring at the surface which destroys the foam stabilising agent as such. This phenomenon may occur with soluble agents[30], and thus is an exception to the general rule that anti-foaming agents must be insoluble in the liquid in which they act.

The anti-foaming agent may be adsorbed on the foaming agent or *vice versa*, reducing the ability of the foaming agent to cause foam. The anti-foaming agent may form insoluble droplets which act as discontinuities and form points of rupture. The droplet considered is, of course, larger than the thickness of the film that would be formed. Difficulties arise in reconciling the agent's disruptive properties with its emulsifiable properties (without which it would not stay in the oil but would form a separate layer), but these can perhaps be overcome in certain circumstances[3].

Some additives which act as anti-foaming agents at certain concentrations will act as foam stabilisers at other concentrations or if water is absent[3]. Further, some agents act as "de-foamers" by causing a foam to collapse; while others act as "anti-foamers" by preventing the bubbles which have risen to the surface from forming a foam. With some "anti-foaming" agents, a small residual foam accumulates which is very persistent —the action thus exhibits inferior "de-foaming" properties. Most agents in practice function as "de-foamers."

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TECHNICAL NOTES

Contributions to this Section of the JOURNAL *will be eligible for Journal Premium Awards and will normally be published within two months of being received.*

Some Blade Designs for an Axial Flow Compressor Stage

by

R. G. TAYLOR

(Department of Mathematics, Bedford College, University of London)

TWO DESIGN CONDITIONS for an axial flow compressor stage are proposed and examined. These are, the constant reaction condition (incorporating "radial equilibrium"), and the condition that the Mach number at inlet to the rotor shall be invariant with radius. In addition, the combination of these two properties in one stage is considered. It is found, with further assumptions regarding the nature of the flow, that a forced vortex type of flow will satisfy both design specifications. The forced vortex solutions for the various cases are presented, and for constant Mach number at inlet to the rotor, more general solutions are given.

1. INTRODUCTION

In axial flow compressor design, the stage which has equal static temperature rises in rotor and stator at all radii has been discussed frequently[1-7]. The naming of this stage has varied slightly, sometimes being called simply "constant reaction," at other times more explicitly "50 per cent. reaction;" the latter is used here.

Two advantageous and quite distinct properties have been claimed for this design. The first is, that by designing for equal diffusion in rotor and stator passages, the losses will probably be minimised, particularly compared with impulse or near impulse blading. The second property concerns the Mach number at inlet to the rotor. Howell[1] comments (p. 447) that "Constant reaction blades tend to have constant Mach numbers along the blade, while these vary considerably for free vortex." A more complete discussion of this property is given by the same author[3], where he extensively discusses 50 per cent. reaction blading, but chooses for detailed analysis a stage which he designates "a modified form of constant reaction blading" as being more suited to his purpose. It is from this modified design that he deduces the near constancy of the Mach number at inlet to the rotor, in strong contrast to a similar free vortex design. (The same feature of the design also applies to the stator inlet Mach number.) Obviously a design with such a characteristic would be very useful when it is desired to work with a high inlet velocity.

Since the near constancy of the Mach number is revealed only as a feature of the particular modifications of 50 per cent. reaction blading considered in Ref. 3, it may not follow from a strictly 50 per cent. reaction design. Thus these two desirable qualities, equal rotor and stator diffusion and a closely limited Mach number variation, may not necessarily be combined in one and the same design. The aim of this paper is to show that a stage can be designed which is a strictly 50 per cent. reaction stage and has a constant Mach number at inlet to the rotor. The result is achieved by designing (*a*) for strictly 50 per cent. reaction, (*b*) for a constant Mach number at inlet to the rotor, and (*c*) for a combination of the two. Even with these design restrictions, the number of possible designs is very large if full use is made of arbitrary radial variations of axial velocity and total head. So, to facilitate comparison with the more conventional types of blade designs, very simple laws will be assumed wherever possible for the behaviour of V_a and H radially. For the radial component of velocity, it does not seem reasonable in a discussion of this simple nature to take it as other than zero, as most authors have done.

The results of this paper show the striking fact that a special case of all three possibilities (*a*), (*b*) and (*c*) can be obtained by a flow in which the whirl velocity is proportional to radius. Such a flow will be called forced vortex, and it is interesting to recall that in 1945 Constant[7] anticipated that this type of flow might have useful attributes.

More general solutions for case (*b*) are obtained, within the design conditions specified, and these are presented in an Appendix but they are not discussed there beyond a few elementary comments.

NOTATION

A	energy input by rotor
C_1, C_2, C_3	arbitrary constants (*see* equations (9), (15), (16)
H	total head
K_p	specific heat at constant pressure
M	Mach number
p	gas pressure
r	radius
R	gas constant
T	temperature
U	peripheral blade speed
v	non-dimensional ratio defined in equation (13)
V_a	axial gas velocity
V_w	whirl gas velocity
γ	ratio of gas specific heats
Γ	degree of reaction
κ	parameter defined in equation (14)
ρ	gas density

Suffixes

0	denotes inlet to the rotor
1	denotes outlet from the rotor

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2. DESIGN CONDITIONS AND ASSUMPTIONS

It is assumed that there is no radial flow, and so the equation of motion in this direction reduces to

$$\frac{V_w^2}{U} = \frac{1}{\rho}\frac{dp}{dU}, \qquad . \quad . \quad (1)$$

where U, the peripheral speed at a given radius, is used as the independent variable instead of r, to simplify the later analysis.

The equation of total head is

$$H = \frac{1}{2}(V_a^2 + V_w^2) + \int \frac{dp}{\rho} \qquad . \quad . \quad (2)$$

and the equation of condition is

$$\frac{p}{\rho^\gamma} = \text{constant}. \qquad . \quad . \quad . \quad (3)$$

The analysis of the air flow for the purpose of designing blade inlet and outlet angles is confined to the region between blade rows, where the suffix "0" denotes conditions at inlet to the rotor and suffix "1" conditions at outlet from the rotor. With this convention the equation for the work done by the rotor is

$$A = U(V_{w1} - V_{w0}). \qquad . \quad . \quad (4)$$

3. EQUATIONS RELATING THE DEGREE OF REACTION TO THE WORK DONE

It is assumed in this section that the velocity head at exit from the stage is the same as at entry. Then the degree of reaction can be defined by[5]

$$\Gamma = \frac{\text{static temperature rise in the rotor}}{\text{stage temperature rise}}.$$

In terms of velocities this becomes

$$2A(1-\Gamma) = V_{a1}^2 - V_{a0}^2 + V_{w1}^2 - V_{w0}^2. \quad . \quad (5)$$

We can eliminate p and ρ from (1), (2) and (3) to give

$$\frac{dH}{dU} = \frac{d}{dU}\left[\tfrac{1}{2}(V_a^2 + V_w^2)\right] + \frac{V_w^2}{U} \quad . \quad . \quad (6)$$

Since $H_1 - H_0 = A$, (5) and (6) give

$$\frac{dA}{dU} = \frac{d}{dU}[A(1-\Gamma)] + \frac{V_{w1}^2 - V_{w0}^2}{U}$$

or

$$\frac{d(A\Gamma)}{dU} = \frac{V_{w1}^2 - V_{w0}^2}{U}. \qquad . \quad . \quad . \quad (7)$$

For some purposes equation (7) is more useful as

$$\frac{d(A\Gamma)}{dU} = \frac{2A(1-\Gamma)}{U} - \frac{V_{a1}^2 - V_{a0}^2}{U}. \quad . \quad (8)$$

4. A CONSTANT REACTION DESIGN

It is apparent from equation (7) that if Γ is to be invariant with radius, A can only be invariant also if

$$V_{w1}^2 = V_{w0}^2.$$

The case $V_{w1} = V_{w0}$ is trivial as it means $A = 0$. If $V_{w1} = -V_{w0}$, the blades will be of impulse shape (but performing quite a different function from that which gives the impulse blade its name) and the whole change in velocity head is made at the expense of varying the axial velocity. If therefore Γ is to be kept constant, this can be done usefully only by allowing the work done to vary radially. Further, V_{a1} can be different from V_{a0}, giving another degree of freedom but, as is more usual in compressor design, we shall design for a constant axial velocity axially. (Note that $V_{a1} = V_{a0}$ does not impose a restriction on the axial velocity profile, except that it shall be the same before and after the rotor. It can vary from the ideal straight line to the roughly parabolic shape suggested by the results of Ref. 6 without violation of this restriction.) Equation (8) with $V_{a1} = V_{a0}$ can readily be solved to give

$$A = C_1 U^{2\left(\frac{1-\Gamma}{\Gamma}\right)} \qquad . \quad . \quad . \quad . \quad (9)$$

where C_1 is an arbitrary constant.

For 50 per cent. reaction $\Gamma = \frac{1}{2}$, and so, combining (4), (5) and (9),

$$\left.\begin{aligned} A &= C_1 U^2 \\ V_{w0} &= \frac{1-C_1}{2}U, \qquad V_{w1} = \frac{1+C_1}{2}U \end{aligned}\right\} \quad . \quad (10)$$

To keep V_{w0} positive, we must have $C_1 < 1$, and so

$$V_{w0} < \tfrac{1}{2}U.$$

Equations (10) show that V_{w0} and V_{w1} are proportional to r, that is, the flow is of the forced vortex type at both inlet to and outlet from the rotor.

5. DESIGNS GIVING A CONSTANT MACH NUMBER AT INLET TO THE ROTOR

If M is the Mach number at rotor inlet, then

$$V_{a0}^2 + (U - V_{w0})^2 = M^2\gamma RT. \qquad . \quad (11)$$

Since equation (1) may be written

$$\frac{V_w^2}{U} = \frac{dT}{dU}K_p,$$

we can eliminate T from (11) and obtain

$$\frac{d}{dU}(U - V_{w0})^2 = M^2\gamma\frac{R}{K_p}\frac{V_{w0}^2}{U} - \frac{d}{dU}(V_{a0}^2). \quad (12)$$

An alternative way of writing equation (12) is found by substituting for V_{a0} from (6), thus giving

$$\frac{1}{2}\frac{d}{dU}(U^2 - 2V_{w0}U) = \frac{V_{w0}^2}{U}\left[\frac{\gamma R}{2K_p}M^2 + 1\right] - \frac{dH_0}{dU}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12a)$$

One solution each, the simplest, at least theoretically, of (12) and (12*a*) will be given. These are the solutions obtained by making V_{a0} and H_0 invariant with radius, in turn. This results in each case in a simple first order homogeneous differential equation in V_{w0} and U. As usual these equations can be solved by writing

$$V_{w0} = vU \qquad . \quad . \quad . \quad (13)$$

where v is a function of U to be found, and the solutions are presented in the Appendix. However, a special solution of this type of differential equation can always be found for which v takes the value of a specific, calculable constant, but one which may be complex.

For, a substitution of type (13) yields only polynomial terms in v, and (dv/dU) to the first degree so that the differential equation becomes

$$f(v)+\phi(v)\frac{dv}{dU}=0$$

where $f(v)$ and $\phi(v)$ are polynomials. Then making v constant simply implies solving the equation

$$f(v)=0.$$

In both cases under discussion the solution of $f(v)=0$ gives real values for v.

For convenience a parameter κ is introduced, where

$$\kappa^2=\frac{M^2\gamma R}{2K_p}=M^2\frac{\gamma-1}{2}. \qquad (14)$$

We are concerned only with subsonic flow and so $\kappa<1$ for all M.

(i) $V_{a0}=constant$

After substitution of (13) into (12), with v and V_{a0} constant, we have

$$(1-v)^2=\kappa^2v^2,$$

that is

$$v=\frac{1}{1\pm\kappa}.$$

In order to have $V_{w0}<U$ (*i.e.* $v<1$), the solution required is

$$v=\frac{1}{1+\kappa}(>\tfrac{1}{2},\text{ for all }\kappa).$$

Thus the forced vortex flow

$$V_{w0}=\frac{U}{1+\kappa},\qquad V_{a0}=\text{constant}$$

gives a constant Mach number at inlet to the rotor. This design has not much practical use since $V_{w0}>\frac{1}{2}U$ means that the rotor inlet angle is too small to be practicable.

(ii) $H_0=constant$

A similar substitution of (13) into (12*a*), this time with v and H_0 constant, gives

$$1-2v=(\kappa^2+1)v^2,$$

whence

$$v=\frac{-1\pm(2+\kappa^2)^{\frac{1}{2}}}{1+\kappa^2}.$$

For positive values of V_{w0} (*i.e.* $v>0$) the solution required is

$$v=\frac{(2+\kappa^2)^{\frac{1}{2}}}{1+\kappa^2}-\frac{1}{1+\kappa^2}(<\tfrac{1}{2}\text{ for all }\kappa).$$

The forced vortex flow in this case is defined by

$$V_{w0}=\left(\frac{(2+\kappa^2)^{\frac{1}{2}}}{1+\kappa^2}-\frac{1}{1+\kappa^2}\right)U,\qquad H_0=\text{constant}.$$

6. A DESIGN COMBINING 50 PER CENT. REACTION AND CONSTANT INLET MACH NUMBER

Referring to the constant reaction design derived in Section 4, in which it was found that

$$V_{w0}<\tfrac{1}{2}U,$$

we see that this can be combined with a constant Mach number design of the second type for all M (*i.e.* all κ). That is, a stage having constant total head radially at inlet to the rotor can be designed to have constant Mach number at inlet to the rotor and to operate at 50 per cent. reaction. The inlet rotor angle is practicable since $V_{w0}<\frac{1}{2}U$, in fact for all M, V_{w0} varies only slightly from $0{\cdot}4U$. It would, of course, give greater flexibility to the design of a 50 per cent. constant M stage if H_0 as well as V_{a0} were to vary radially, but in that case conditions at entry to the stage would be very difficult to achieve.

7. USE OF THESE SPECIAL DESIGNS

Appendix II of Ref. 3 shows that in a multi-stage compressor the work done per stage tends to a value which is independent of the radius, and that in most compressors this steady state may be closely realised after only two or three stages. This does not preclude the use of a single stage which is designed for varying work done and required to fulfil some particular purpose in a multi-stage compressor[1]. For instance, in a compressor working with a high inlet Mach number it would be useful to be able to design the first stage to have a constant or near constant Mach number radially even at the expense of varying work done radially. Essentially the designs given in this note, and their extensions in the Appendix, apply to some such specific use. They are not proposed here as suitable for the overall design of a multi-stage compressor. In the case of the 50 per cent. reaction design (or a design having any constant degree of reaction except $\Gamma=1$) it was seen that, with the restriction $V_{a1}=V_{a0}$, a varying work done was inevitable. The designs for constant Mach number at inlet to the rotor make no mention of the work done, since no restriction on it is implicit in the design specification. A stage with constant work done radially and constant rotor inlet Mach number can obviously be achieved simply by designing so that

$$V_{w1}=\frac{A}{U}+V_{w0}$$

where A is the constant work done. This will mean, however, that V_{a1}, is completely determined, given V_{a0}, and different from V_{a0}. So again, radial variation of work done may be desirable to increase the flexibility of the design in other directions.

APPENDIX

With V_{a0} invariant radially, the complete solution to equation (12) is

$$\left(\frac{C_2}{U}\right)^{2(1-\kappa^2)}=\{v(1+\kappa)-1\}^{1-\kappa}\{v(1-\kappa)-1\}^{1+\kappa}. \qquad (15)$$

where C_2 is an arbitrary constant and $v=V_{w0}/U$. It can easily be seen that the forced vortex flow solution corresponds to

$$C_2=0.$$

The graph of equation (15) for positive values of v and U consists of two parts. Starting from zero at

$U/C_2=1$, it increases monotonically to approach the value $1/(1+\kappa)$ asymptotically. The other part of the curve is of no particular interest since it is for values of $v>1$.

With H_0 invariant radially, the complete solution to equation (12*a*) is

$$v=\frac{(2+\kappa^2)^{\frac{1}{2}}-1}{1+\kappa^2}+\frac{(2+\kappa^2)^{\frac{1}{2}}}{1+\kappa^2}\,\frac{2}{(U/C_3)^{2\sqrt{(2+\kappa^2)}}-1}. \qquad (16)$$

where C_3 is an arbitrary constant and $v=V_{w0}/U$. The solution giving a forced vortex flow is again given by taking the value of the arbitrary constant to be zero.

For positive values of v and U, the graph of (16) consists of a single monotonic curve asymptotic to the two lines

$$\frac{U}{C_3}=1 \qquad \text{and} \qquad v=\frac{(2+\kappa^2)^{\frac{1}{2}}-1}{1+\kappa^2}.$$

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The Flexural Centre or Centre of Shear

by

W. T. KOITER, A.F.R.Ae.S.

(*Professor of Applied Mechanics, Delft University*)

IN view of the recent notes in this JOURNAL by Professors Jacobs[1] and Duncan[2], it may be of interest to give some additional references to investigations of the flexural centre of beams, which otherwise might be overlooked, and to add some remarks which seem to have escaped the attention of previous writers.

Although the earliest investigation seems to have been published by Griffith and Taylor (Ref. 4 of Duncan's paper), Eggenschwyler[3] gave an independent discussion of the problem for narrow beams. Considerable progress was made by Weber[4,5] in two papers, published in 1924 and 1926. Similar results were obtained by Schwalbe[6], who was apparently unaware of Weber's previous investigations. Trefftz (Ref. 6 of the paper by Jacobs) confirmed Weber's and Schwalbe's results, starting from the definition that the centre of shear is the point where a transverse load should be applied in order to make the strain energy a minimum. Following Jacobs and Duncan this point will be called the centre of least strain.

Meanwhile van der Neut[7] had given a simplified method for the determination of the flexural centre for thin-walled multi-cell box beams, but Leibenson[8] showed later that van der Neut's basic equation for the line-integral of the shear stress along the contour of any cell

$$\int \tau_s ds=0 \qquad . \quad . \quad . \qquad (1)$$

in pure bending and shear, did not agree with the classical theory of flexure by de Saint-Vénant.

This discrepancy prompted a rigorous investigation by the present writer in 1937, which was published in 1939[9]. In this paper the definition of the flexural centre as the point of load application for zero mean twist was adopted and the following formulae for its position were obtained for a solid section

$$\left.\begin{aligned} x_0&=\frac{1}{I_x}\left[\int\int -y\phi\,dxdy+\frac{\sigma}{1+\sigma}\int\int xF\,dxdy\right],\\ y_0&=\frac{1}{I_y}\left[\int\int x\phi\,dxdy+\frac{\sigma}{1+\sigma}\int\int yF\,dxdy\right], \end{aligned}\right\} \qquad (2)$$

The x and y axes are the principal axes of the cross section, ϕ is the torsional warping function, defined by the equation $\nabla^2\phi=0$ with boundary condition

$$\frac{\partial\phi}{\partial n}=y\cos(x,n)-x\cos(y,n),$$

F is Prandtl's stress function (χ in Jacobs' paper and ψ in Duncan's paper), defined by the equation $\nabla^2F=-2$, with boundary condition $F=0$, and σ is Poisson's ratio. It is easily verified that these formulae agree with Jacobs' formulae (2) and it can also be shown that they agree with Duncan's formula (5) for a symmetric section.

For a multiply connected cross section Prandtl's stress function F is again determined by the equation $\nabla^2F=-2$, but now with the boundary conditions $F=0$ on the outer contour and $F=$constant on each inner boundary[10]. It was shown by the writer[9] that the formulae (2) are now again applicable if the *second* integrals are taken over the *entire area enclosed by the outer contour* and the value of F inside a cavity is taken equal to its value on the boundary of the cavity.

It has sometimes been argued that the centre of least strain is more useful than the flexural centre, because the strain energy due to bending with shear by a load in the centre of least strain and the strain energy due to a torsional moment with respect to the axis of least

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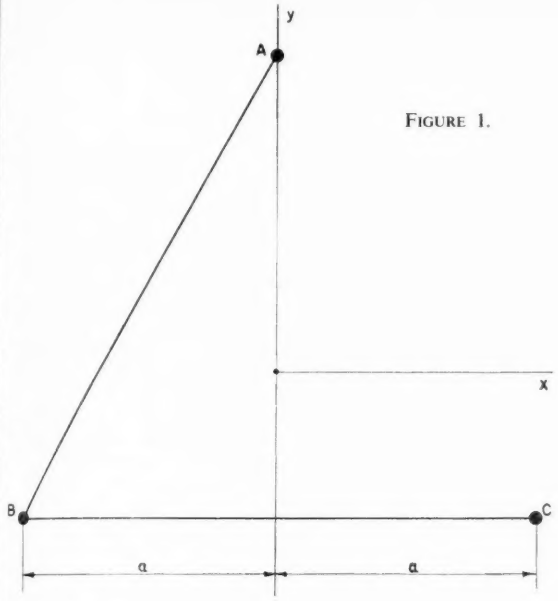

FIGURE 1.

strain are additive. However, the value of this argument should not be over-estimated because the energies due to bending with shear by two loads acting in the centre of least strain in the directions of the two principal axes are *not* additive. This is easily seen in the example of Fig. 1, a cross section of a beam with three equal (heavy) flanges and two (thin) shear webs. The flexural centre and the centre of least strain coincide in *B*. A load parallel to the *y*-axis induces shear flows in both webs, whereas a load in the direction of the *x*-axis causes a shear flow only in web *BC*. The strain energy therefore has a coupling term arising from the shear energy in web *BC*.

On the other hand, the definition of the flexural centre has the slight advantage that it also applies to a beam under *uniformly distributed* loads. This property was stated by the writer in one of the proposals accompanying his doctor's thesis[11]. It is proved by observing that the local rate of twist can be expressed in the shear strains e_{zx} and e_{zy}

$$\frac{\partial w_z}{\partial z} = \frac{1}{2}\left(\frac{\partial e_{zy}}{\partial x} - \frac{\partial e_{zx}}{\partial y}\right) \quad . \quad . \quad (3)$$

and by making use of the fact that the shear stress distribution in the cross section for a beam under uniform load is *exactly similar* to the shear stress distribution for a beam under terminal load[12]. A similar property has apparently not yet been proved (and possibly does not always hold) for the centre of least strain.

Finally, the author may perhaps be allowed to express his surprise at the large difference between the positions of the flexural centre and the centre of least strain in the aerofoil section analysed by Jacobs[1].

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The Noise From Aircraft

by
A. G. WALTERS, Ph.D.
(*Armament Research Establishment*)

A GENERAL EXPRESSION is obtained for the potential of sound or aerodynamic disturbance from a source moving with variable vector velocity, based on linear theory. Knowing the space time curve for the source, the Doppler frequency corrections and intensities can be determined. If in any part of its trajectory the aircraft exceeds the velocity of sound, the number of sonic bangs and the times of occurrence can be calculated. A numerical example is given. The following conclusions are deduced: (*i*) If at any point of its trajectory, the component of the aircraft velocity in the direction of the observer is equal to the velocity of sound, a sonic bang is propagated to the observer with the velocity of sound. This has been previously postulated by C. H. E. Warren. (*ii*) A bang as determined by conclusion (*i*) comprises two components and in certain circumstances the components in some of the bangs may be acoustically separable, thus giving rise to double bangs.

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NOTATION

ϕ	potential of sound or aerodynamic potential
c	velocity of sound
x, y, z	Cartesian co-ordinates specifying position of observer
x_1, y_1, z_1	Cartesian co-ordinates specifying position of source
R	defined by equation (2)
τ	time at which pulse is emitted from source
t	time with reference to observer
r	radius of small sphere
K	a constant specifying strength of elementary source
ε	small value of time
α, β	co-ordinates of observer relative to Q (Figs. 3 and 4)
V	velocity of source at time τ
L_2	defined by equation (6)
L_1	defined by equation (10)
δ_1	defined by equation (8)
δ_2	defined by equation (7)
δ_3	defined by equation (10)
X_1	defined by equation (11)
θ	angle between vector from source to observer and vector velocity of source
$p/2\pi$	frequency of a wave emitted by source
$F(\tau)$	a function expressing the strength of the source (sound or aerodynamic)
I	relative intensity of sound as observed (equation (14))

The propagation of sound waves from a source moving with constant vector velocity has been discussed in earlier papers[1, 2]. In this note a general formula is derived for determining the intensity of sound and the Dopplerised frequencies heard by an observer at rest when the source moves with a non-uniform velocity.

The differential equation for the propagation of sound waves in three dimensions is of the form

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}, \quad . \quad . \quad . \quad . \quad (1)$$

c being the velocity of sound. A solution of this is

$$\left.\begin{aligned} \phi &= \frac{F(t-R/c)}{4\pi Rc^2}, \quad t > R/c, \\ &= 0 \qquad\qquad , \quad t < R/c, \end{aligned}\right\} \quad . \quad (2)$$

where $R^2 = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2$,

and $F(t)$ is any function of t, not necessarily continuous.

In general the function expressed by equation (2) diverges when the point x, y, z coincides with x_1, y_1, z_1 for all values of t and it is easily seen that the function ϕ as expressed by equation (2) is the potential at the point x, y, z at time t due to a point source at x_1, y_1, z_1 which begins at time $t=0$. Constructing a small sphere of radius r and centre at the point x_1, y_1, z_1 the potential flow over the surface of this sphere per unit time is

$$-4\pi r^2\left(\frac{\partial\phi}{\partial R}\right)_{R=r} = \frac{F(t)}{c^2},$$

and this defines the strength of the source.

Suppose that $F(t)$ is of the form

$$\begin{aligned} F(t) &= 0, \quad 0 < t < \tau, \\ &= K, \quad \tau < t < \tau + \varepsilon, \\ &= 0, \quad \tau + \varepsilon < t, \end{aligned}$$

K being a constant and τ and $\tau+\varepsilon$ being some values between 0 and t. The corresponding value ϕ is thus

$$\begin{aligned} \phi &= 0, \qquad\qquad 0 < t - R/c < \tau, \\ &= K/(4\pi Rc^2), \quad \tau < t - R/c < \tau + \varepsilon, \\ &= 0, \qquad\qquad \tau + \varepsilon < t - R/c. \end{aligned}$$

This is the potential at x, y, z at time t due to a pulse emitted between the times τ and $\tau+\varepsilon$ at the point x_1, y_1, z_1. It is seen that inside the sphere of radius $c(t-\tau-\varepsilon)$ and centre at x_1, y_1, z_1, the potential ϕ is zero and outside the sphere of radius $c(t-\tau)$, the potential is also zero. In the region bounded by these two spheres, the potential is given by

$$\phi = \frac{K}{4\pi Rc^2}. \quad . \quad . \quad . \quad . \quad (3)$$

This is true for all values of ε however small provided that $\tau+\varepsilon$ is less than t. Thus the pulse moves outwards as a spherical shell of thickness $c\varepsilon$. A continuous source can be considered as emitting continuously pulses of this kind and using this concept, the potential due to a moving source can be obtained in a general form.

The source begins at time $t=0$ at the point A and moves with varying velocity along the curve AB (Fig. 1), the observer being at P.

Suppose that the source is at Q at time τ at which time it emits a pulse which reaches P at time t. Thus

$$PQ = c(t-\tau),$$

or

$$t = \tau + PQ/c. \quad . \quad . \quad . \quad (4)$$

If the positive distance PQ is known for all values of τ, the value of t given by equation (4) can be plotted against τ. It appears as in Fig. 2.

The curve may have maxima and minima at times t given by $t_0, t_1, t_2, \ldots\ldots$ For a given value of t, there may be zero, one, two or more corresponding values of τ as obtained from the graph; let these values of τ be $\tau_a\, \tau_b\, \tau_d \ldots\ldots$, all of which we suppose are positive and are less than t (since PQ are taken as positive always). This means that the pulses which are emitted by the source at times $\tau_a, \tau_b, \tau_d \ldots\ldots$, arrive at P

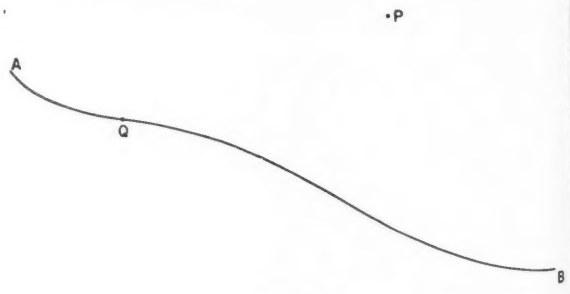

FIGURE 1

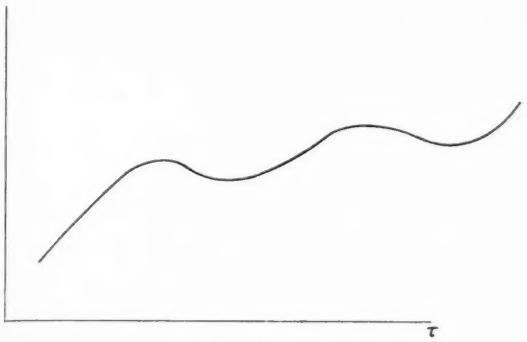

FIGURE 2.

simultaneously at the time t. If, for example, the source emits a simple wave, the number of waves heard by the observer simultaneously at time t is the number of values of τ corresponding to t. It is shown later that all the waves have different frequencies and it is also shown that the sonic bangs are heard by the observer at times $t_0, t_1, t_2 \ldots\ldots$

In the derivation of equation (3), it has been assumed that the source remains at rest. When the source has a velocity comparable with the velocity of sound, the result must be modified. To do this some results given in references 1 and 2 are used. Suppose that the source is at Q at time τ and at a time $\tau+\varepsilon$ it is at E. Between these times assume the velocity V to be constant (Fig. 3). In the first case V is taken to be greater than the velocity of sound c.

After time t the pulse emitted at Q has moved out to a spherical surface centre Q and radius $c(t-\tau)$ and that emitted at E has moved out to the spherical surface centre E and radius $c(t-\tau-\varepsilon)$. The surfaces intersect along the circle, the plane of which is perpendicular to QE and which intersects the plane of the paper at e and g. In Refs. 1 and 2 it is shown that, outside the two spherical surfaces, ϕ is zero; also, in that region which is inside both spheres, ϕ is zero. Inside the two regions which are contained in one of the spheres, but outside the other, ϕ is given by equation (5).

$$\phi=\frac{F(\delta)}{4\pi c(V^2-c^2)^{\frac{1}{2}}L_2}, \quad . \quad . \quad (5)$$

where
$$L_2^{\,2}=-\beta^2+\frac{c^2X_1^{\,2}}{(V^2-c^2)} \quad . \quad . \quad (6)$$

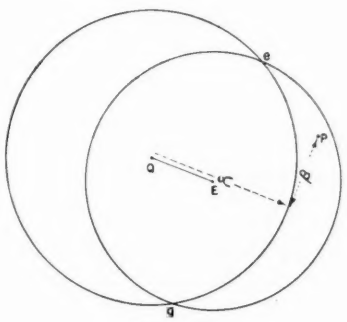

FIGURE 3.

In the region forward of the circle of intersection

$$\delta\equiv\delta_2=t-\frac{X_1V}{V^2-c^2}-\frac{L_2}{(V^2-c^2)^{\frac{1}{2}}}, \quad . \quad (7)$$

and in the region behind the circle of intersection

$$\delta\equiv\delta_1=t-\frac{X_1V}{V^2-c^2}+\frac{L_2}{(V^2-c^2)^{\frac{1}{2}}}. \quad . \quad (8)$$

The distances X_1, β used in these are obtained as follows: β is the perpendicular distance from P on to QE and α is the distance from P to Q measured parallel to QE. Then $X_1=V(t-\tau)-\alpha$. When the distance QE is very small it is easily seen that the angle EQe is $\cos^{-1}(c/V)$.

When V is less than c the picture appears as in Fig. 4. The spherical surfaces do not intersect and in the region included between the two spherical surfaces the potential ϕ is given by equation (9) and elsewhere it is zero,

$$\phi=\frac{F(\delta_3)}{4\pi c(c^2-V^2)^{\frac{1}{2}}L_1}, \quad . \quad . \quad (9)$$

where
$$\delta_3=t+\frac{X_1V}{c^2-V^2}-\frac{L_1}{(c^2-V^2)^{\frac{1}{2}}}, \quad . \quad (10)$$

$$L_1^{\,2}=\beta^2+\frac{c^2X_1^{\,2}}{(c^2-V^2)},$$

where, as before,

$$X_1=V(t-\tau)-\alpha. \quad . \quad . \quad . \quad (11)$$

Suppose QE is very small. From Fig. 4

$$c^2(t-\tau)^2=\alpha^2+\beta^2.$$

Writing $\alpha=V(t-\tau)-X_1$, we find

$$(V^2-c^2)(t-\tau)^2-2VX_1(t-\tau)+\beta^2+X_1^{\,2}=0,$$

or
$$t-\tau=\frac{VX_1}{V^2-c^2}\pm\frac{L_2}{(V^2-c^2)^{\frac{1}{2}}}. \quad . \quad (12)$$

Thus both δ_2 and δ_1 are equal to τ. Similarly, taking V less than c we find δ_3 is equal to τ.

To apply this theory to the general aircraft motion given by Fig. 1, proceed thus: For a given value of t determine the values of τ from Fig. 2. Let these be $\tau_a, \tau_b, \tau_d \ldots\ldots$ First consider the pulse from τ_a. At time τ_a the aircraft is at Q_a and $PQ_a=c(t-\tau_a)$. Let the small element of trajectory at Q_a be along Q_aE (*i.e.* the velocity at Q_a is in this direction). Let the angle EQ_aP be θ_a.

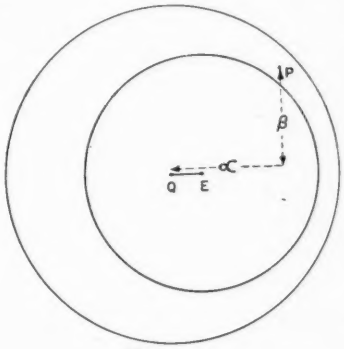

FIGURE 4.

Thus

$$\alpha = c\,(t-\tau_a)\cos\theta_a, \qquad \beta = c\,(t-\tau_a)\sin\theta_a.$$

Let the velocity at Q_a be V_a

(*i*) If $V_a < c$, the potential at P due to this pulse is (equation (9))

$$\phi_a = \frac{F(\tau_a)}{4\pi c\,(c^2 - V_a^2)^{\frac{1}{2}}\,L},$$

with the above values of α and β in L_1.

(*ii*) If $V_a > c$

$$\phi_a = \frac{F(\tau_a)}{4\pi c\,(V_a^2 - c^2)^{\frac{1}{2}}\,L_2},$$

with the foregoing values of α and β in L_2.

From equation (12)

$$\begin{aligned}\pm(V^2-c^2)^{\frac{1}{2}}L_2 &= (V^2-c^2)(t-\tau) - VX_1,\\ &= (V^2-c^2)(t-\tau) - V^2(t-\tau) + V\alpha,\\ &= (V^2-c^2)(t-\tau) - V^2(t-\tau) + \\ &\qquad + Vc(t-\tau)\cos\theta,\\ &= c(t-\tau)[V\cos\theta - c].\end{aligned}$$

The denominator for ϕ is positive, as can be seen from equation (2) and thus for $V_a > c$

$$\phi_a = -\frac{F(\tau_a)}{4\pi c^2\,(t-\tau_a)\,(V_a\cos\theta_a - c)}. \qquad (13)$$

It can be shown that the same formula applies when $V_a < c$. The potentials from the pulses emitted at $\tau_b, \tau_d \ldots\ldots$ are determined from the same formula with $\tau_b, \tau_d \ldots\ldots$ replacing τ_a and the total potential at P is thus

$$\phi = \phi_a + \phi_b + \phi_d \ldots\ldots$$

If the source emits a simple sine wave continuously of the form $A\sin pt$ the potential of sound at P at time t is

$$\phi = -\frac{A\sin p\tau_a}{4\pi c^2\,(t-\tau_a)(V_a\cos\phi_a - c)} - \frac{A\sin p\tau_b}{4\pi c^2\,(t-\tau_b)\,(V_b\cos\phi_b - c)} + \ldots\ldots$$

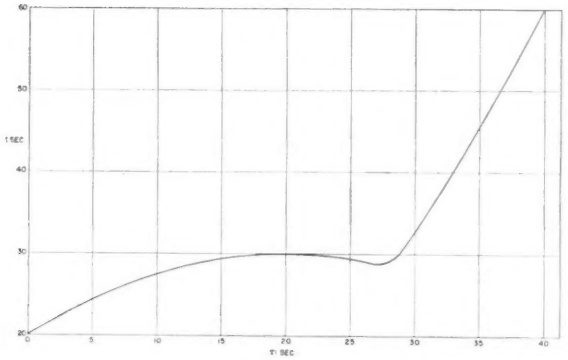

FIGURE 5.

Thus the observer hears a number of distinct sound waves simultaneously, the number being equal to the number of values of τ. The frequencies are

$$\frac{p}{2\pi}\left|\frac{d\tau}{dt}\right|_a, \quad \frac{p}{2\pi}\left|\frac{d\tau}{dt}\right|_b, \ldots\ldots$$

The energy of these sound waves is

$$I = \left(\frac{A}{4\pi c^2}\right)^2 \left\{ \frac{\left(\frac{dt}{d\tau}\right)_a^2}{(t-\tau_a)^2(V_a\cos\theta_a - c)^2} + \frac{\left(\frac{d\tau}{dt}\right)_b^2}{(t-\tau_b)^2(V_b\cos\theta_b - c)^2} + \ldots\ldots \right\}.$$

The aircraft as a source of sound is a complex process, the sound arising from the engines and the aerodynamic drag among other things. The whole probably forms a wide spectrum. The shape and intensity of this spectrum depend on the velocity and acceleration. If the noise emitted is considered as a series of sine waves, the amplitude factors A depend on the frequency and the value of τ. Thus A is replaced in the expression for the intensity by the values of $A(\tau)$ inside the brackets and sum over the frequencies covering the spectrum. The value of $A(\tau)$ is not known and for the purposes of this work it is taken as a constant. The Doppler correction factors $d\tau/dt$ are independent of p and are thus the same for all frequencies emitted. The relative intensity of sound (for comparing intensities at different points and different times) may thus be taken as

$$I = \frac{\left(\frac{d\tau}{dt}\right)_a^2}{(t-\tau_a)^2(V_a\cos\theta_a - c)^2} + \frac{\left(\frac{d\tau}{dt}\right)_b^2}{(t-\tau_b)^2(V_b\cos\theta_b - c)^2} \ldots\ldots, \qquad (14)$$

that is, the intensity distribution with frequency of the emitted spectrum is taken as independent of the time.

Finally, the basis of these results is the differential equation of the propagation of sound (equation (1)). It is shown in Ref. 1 that this is the same as the non-steady linear differential equation of aerodynamics and thus we may state that the theory is based on the linear theory of aerodynamics.

Example 1

The aircraft starting at A is assumed to travel in a straight line with constant velocity V. The observer is at the point P at a distance x from A measured parallel to the line of flight and ρ measured perpendicular to this line (Fig. 6).

When the aircraft is at Q (at time τ) it emits a pulse which reaches P at time t thus

$$(PQ)^2 = (x - V\tau)^2 + \rho^2 = c^2(t-\tau)^2.$$

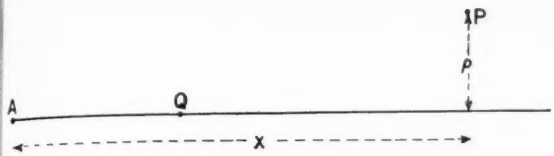

FIGURE 6.

This is a quadratic in τ and thus there are two values of τ. Writing $x=Vt-X$ we find

$$\tau=t-\frac{XV}{V^2-c^2}\pm\frac{L^2}{(V^2-c^2)^{\frac{1}{2}}},$$

where $$L_2^2=\frac{c^2X^2}{(V^2-c^2)-\rho^2}.$$

When V is less than c it is easily seen that only one of these values of τ is in the range $0<\tau<t$, the value being τ_1, where

$$\tau_1=t+\frac{XV}{c^2-V^2}-\frac{1}{(c^2-V^2)^{\frac{1}{2}}}\left[\frac{c^2X^2}{c^2-V^2}+\rho^2\right]^{\frac{1}{2}}.$$

Substituting this value in equation (9) the potential ϕ is obtained. This value τ_1 is only permissible if P is inside the sphere centre A and radius ct. Outside the sphere no value of τ exists in the range $0<\tau<t$.

If V is greater than c, either two values, one value or no value of τ in the range $0<\tau<t$ exist, depending on the position of P and t. This example is worked out in detail in Refs. 1 and 2.

Example 2

The aircraft starts from A with velocity u and moves with constant acceleration f in a straight line.

After time τ the aircraft is at Q where

$$AQ=u\tau+\tfrac{1}{2}f\tau^2,$$

thus

$$(PQ)^2=(x-u\tau-\tfrac{1}{2}f\tau^2)^2+\rho^2,$$

and if the pulse emitted at time τ arrives at P at time t

$$(x-u\tau-\tfrac{1}{2}f\tau^2)^2+\rho^2=c^2(t-\tau)^2.$$

Since τ is less than t

$$t=\frac{\tau+[(x-u\tau-\tfrac{1}{2}f\tau^2)^2+\rho^2]^{\frac{1}{2}}}{c}. \qquad (15)$$

This is a quartic equation for τ and thus there are four values of τ. Possibly one or more of these roots may be negative and thus not a permissible value. The roots can be determined quite readily numerically. Substituting values of τ in equation (15) the corresponding values of t are obtained. Plotting these graphically the curve t against τ is obtained for given values of τ, as in Fig. 5. For a given value of t the values of τ may be read off. Substituting these values into equation (13) the potential ϕ is obtained. Let these values of τ be $\tau_a, \tau_b \ldots\ldots$ then if the frequency of the note emitted by the aircraft is $p/2\pi$ the frequencies of the notes heard by the observer at time t are

$$\frac{p}{2\pi}\left|\frac{d\tau}{dt}\right|_a,\quad \frac{p}{2\pi}\left|\frac{d\tau}{dt}\right|_b \ldots\ldots$$

From equation (15) we find

$$\frac{dt}{d\tau}=1-\frac{(x-u\tau-\tfrac{1}{2}f\tau^2)f\tau}{c[(x-u\tau-\tfrac{1}{2}f\tau^2)^2+\rho^2]^{\frac{1}{2}}}$$

or

$$\frac{d\tau}{dt}=\frac{c^2(t-\tau)}{c^2(t-\tau)-(x-u\tau-\tfrac{1}{2}f\tau^2)f\tau}. \qquad (16)$$

Substituting in this the value of t and $\tau_a, \tau_b \ldots\ldots$ respectively the required frequencies are obtained.

At time τ the aircraft velocity and the value of $\cos\theta$ are

$$V=f\tau+u, \qquad \cos\theta=\frac{x-u\tau-\tfrac{1}{2}f\tau^2}{c(t-\tau)}.$$

Thus the term

$$\frac{\left(\frac{d\tau}{dt}\right)^2}{(t-\tau)^2(V\cos\theta-c)^2},$$

can be calculated for the given value of t and knowing the corresponding value $\tau_a, \tau_b \ldots\ldots$, so that the relative intensity can be calculated from equation (14).

A numerical example has been worked out with the following values: $u=0$, $f=55$ ft./sec.^{2}, $c=1{,}100$ ft./sec., $x=22{,}000$ ft. and $\rho=1{,}100$ ft. With these values equation (15) becomes

$$t=\tau+[(20-0{\cdot}025\tau^2)^2+1]^{\frac{1}{2}},$$

and curve t against τ is plotted in Fig. 5.

The observer hears no sound for 20·02 secs. The curve t against τ has a maximum at $\tau=20$ secs. and a minimum at $\tau=27{\cdot}45$ secs., the values of t being 30·05 secs. and 29·02 secs., respectively. Before 29·02 secs. he hears only the sound generated in the subsonic part of the trajectory. Between 29·02 secs. and 30·05 secs., he hears simultaneously the sounds generated at

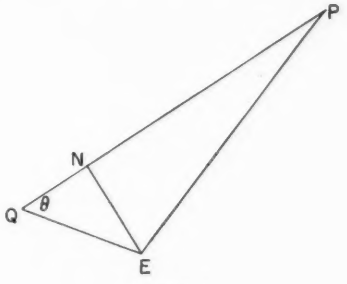

FIGURE 7.

three parts of the trajectory. For example, if the aircraft emitted a note of one frequency continuously, he would hear simultaneously three notes of different frequencies all changing rapidly with time. After 30·05 secs., he hears the sound generated in the supersonic part of the trajectory only. Taking a wave emitted by the aircraft to be of frequency 500 c./s., the frequencies heard by the observer are given in Fig. 8.

Some typical results are given in Table I, where $|V\cos\theta-c|$ is denoted by k.

The relative intensity heard by the observer is given graphically in Fig. 9. At $t=29{\cdot}02$ secs., it is seen that

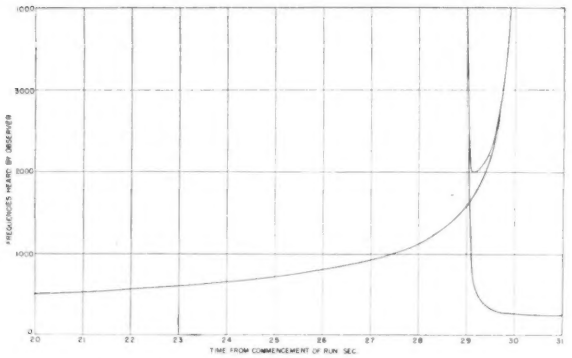

FIGURE 8.

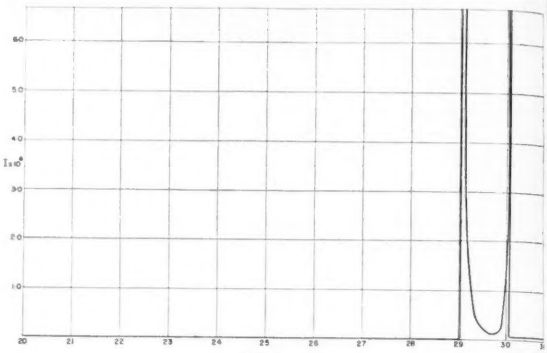

FIGURE 9.

$\left|\frac{d\tau}{dt}\right|_b$ and $\left|\frac{d\tau}{dt}\right|_d$ are infinite, also k_b and k_d are zero at this time and two terms of equation (14) are both infinite, both containing a fourth order infinity. Also at $t=30{\cdot}05$ secs., $\left|\frac{d\tau}{dt}\right|_b$ and $\left|\frac{d\tau}{dt}\right|_a$ are infinite and k_a and k_b are zero, so again two terms of the relative intensity are infinite, both fourth order infinities. Clearly the observer hears the sonic bangs at $t=29{\cdot}02$ and $t=30{\cdot}05$ secs.

GENERAL DISCUSSION

It is seen from equation (14) that if the value of $d\tau/dt$ is infinite, the intensity is also infinite. Also if the value $V\cos\theta=c$, the intensity is infinite again. It is shown now that these conditions are identical. The term $d\tau/dt$ is infinite when t as determined from equation (4), has a stationary value, *i.e.* $dt/d\tau=0$. From equation (4) we have

$$\frac{dt}{d\tau}=1+\frac{1}{c}\frac{d(PQ)}{d\tau},$$

and thus for a stationary value of

$$\frac{d(PQ)}{d\tau}=-c.$$

In Fig. 7, Q is the position of the aircraft at time τ and E the position at time $\tau+d\tau$. This $QE=Vd\tau$. EN is the perpendicular from E on to QP and for very small values of $d\tau$, $PE=PN$ to the first order and $QN=-d(PQ)$. We find

$$-\frac{d(PQ)}{Vd\tau}=\frac{QN}{QE}=\cos\theta,$$

or

$$\frac{d(PQ)}{d\tau}=-V\cos\theta,$$

and thus at a stationary value of t, $V\cos=c$. Thus the intensity of each sonic bang contains a fourth order infinity, as can be seen from equation (14).

Further, at a stationary value of t, two of the terms $\tau_a, \tau_b, \tau_d \ldots\ldots$, become equal. In the foregoing example at $t=29{\cdot}02$ secs., $\tau_a=\tau_b$ and at $t=30{\cdot}5$ secs., $\tau_b=\tau_d$. Thus a bang comprises two terms both of which contain fourth order infinities. This may be of some physical importance. In Fig. 8 it is seen that just beyond $t=29{\cdot}02$ secs., the frequency of one of the pulse systems decreases much more rapidly than the other and thus reaches the audible range first and if the interval is

TABLE I

t	τ_a	τ_b	τ_d	$\left\|\frac{d\tau}{dt}\right\|_a$	$\left\|\frac{d\tau}{dt}\right\|_b$	$\left\|\frac{d\tau}{dt}\right\|_d$	k_a	k_b	k_d
20·02	0	—	—	1·00	—	—	1050	—	—
23	3·31	—	—	1·20	—	—	918	—	—
27	9	—	—	1·82	—	—	606	—	—
29	13·69	—	—	3·17	—	—	347	—	—
29·02	13·74	27·45	27·45	3·20	∞	∞	345	0	0
29·1	13·96	26·48	28·07	3·30	4·02	1·72	333	274	639
29·2	14·28	26·12	28·24	3·48	4·02	1·10	316	274	999
29·3	14·63	25·73	28·38	3·71	4·11	0·835	296	267	1317
29·4	15	25·33	28·48	3·98	4·30	0·706	276	255	1558
29·5	15·46	24·84	28·56	4·38	4·56	0·632	251	241	1739
29·6	15·89	24·5	28·63	4·81	5·01	0·584	228	219	1882
29·7	16·37	24·1	28·68	5·40	5·58	0·563	202	197	1956
29·8	16·97	23·64	28 72	6·50	6·42	0·551	169	172	1993
29·9	17·65	23·0	28·76	8·32	7·74	0·540	132	142	2027
30·0	18·55	21·62	28·80	13·02	13·08	0·531	84	79	2072
30·05	20	20	28·83	∞	∞	0·527	0	0	2099
31	—	—	29·20	—	—		—	—	2107
35	—	—	31·0	—	—		—	—	2198

Note: $|V\cos\theta=c|$ is denoted by k.

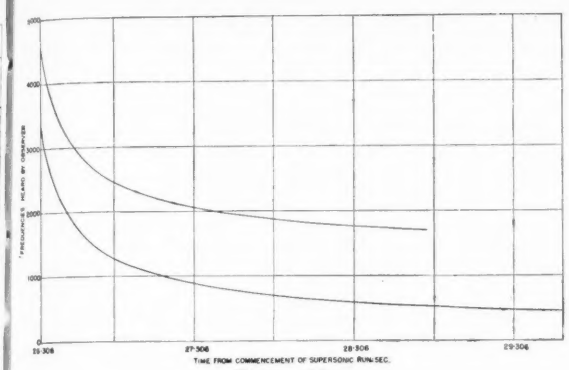

FIGURE 10.

sufficient the two components may be acoustically separable and thus this bang may appear as a double bang. The same discussion applies to the second bang at 30·05 secs. Furthermore the observer is assumed at rest and thus the bangs are propagated as high frequency pulses. Viscosity and heat conduction introduce very large damping factors when the pulse frequency is very high. If the bang is again considered at 29·02 secs., viscosity and heat conduction considerably modify the relative intensity immediately following 29·02 secs., *i.e.* the first part of the two components is heavily damped leaving possibly the lower frequencies, thus the first component is effective before the second due to the more rapid reduction in frequency, thus enhancing the possibility of this appearing as a double bang.

In the example quoted the time intervals are probably too small for the bangs to appear as two double bangs. This is because the observer is too close to the trajectory. To illustrate this the frequencies heard by an observer in the following circumstances are calculated. The aircraft starting at *A* travels with a constant velocity of 1,500 ft./sec. in a straight line. The observer is 30,000 ft. from *A* measured parallel to the trajectory and 10,000 ft. measured perpendicularly. The aircraft emits a note frequency of 500 c./sec. It is seen that the Doppler frequencies reduce relatively slowly and it is possible that this bang may appear as a double bang (*see* Fig. 10).

Thus our conclusions are:

(*i*) If at any point of its trajectory, the component of the aircraft velocity in the direction of the observer is equal to the velocity of sound, a sonic bang is propagated from this point towards the observer with the velocity of sound. This conclusion has been postulated by C. H. E. Warren[3] who derived it on a physical basis.

(*ii*) A bang as determined by conclusion (*i*) comprises two components and in certain circumstances they may be acoustically separable, thus giving rise to a double bang.

In Ref. 2, where it is assumed that the aircraft had a constant vector velocity, it is shown that the bang occurs when the Mach cone passes over the observer. It is easily shown that the time of the bang calculated by this method is identical with that deduced on the basis of conclusion (*i*). After the second bang in the foregoing example only one spectrum is heard by the observer, the observer being near the start of the supersonic run, so that the interval during which the second spectrum is heard is small.

ACKNOWLEDGMENTS

The author is indebted to the Chief Scientist, Ministry of Supply, for permission to publish this article, and to Miss Enid Wells for assistance in the numerical work.

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THE LIBRARY
Reviews

GAS TURBINE ANALYSIS AND PRACTICE. *Burgess H. Jennings and Willard L. Rogers.* 487 *pp. Illustrated. McGraw-Hill,* 1953. 68*s. net.*

American books on this subject generally have three disadvantages from the English reader's viewpoint:—

(1) So many books have already been published that only a completely new approach is worth while. (2) They are very expensive, and (3) they appear deliberately to ignore the important contribution made by British engineers, particularly in the aircraft field. No doubt British books err equally in the opposite way, which probably affects their sales in America.

The present book overcomes none of the disadvantages, though it is only fair to say that the selection of engines for description is so odd that one wonders if the authors are properly acquainted with present levels of development. The authors are partly aware of the first disadvantage, because in their preface they state that when they first began there were few texts, but since then several have appeared. Their hope, that in spite of this their book is still merited, is in this reviewer's opinion quite unjustified.

The first half of the book follows the well-worn line—Thermodynamics and Air Properties—Gas Turbine Cycles—Gas Turbine Types—Axial Flow Compressors—Centrifugal Compressors—Aircraft Propulsion and Jet Engines. It is recommended for the future that books on this subject should assume that the student knows the necessary thermo- and fluid dynamics and save some 100 pages for the title page subject. In particular, 30 of these 100 pages are used up in tables of the thermodynamic properties of air—dull as the telephone directory, considerably less useful and surely, if required, available elsewhere. In such a standard of book it is better to assume fixed specific heats for the separate processes of compression, combustion and expansion—this and the gas laws will give a much better grasp of the subject—with a brief indication of how to use enthalpy charts if such use is justified.

These wasted pages could well have been adapted to better explanations of physical facts than for instance that on p. 83 of the reason why diffusing flow tends to be less efficient than accelerated flow. Again in the description of the closed cycle system the advantages of the system are inadequately explained, the reader being left with the unstated paradox that whereas for a given power output the engine size is reduced, the Reynolds numbers are favourably high. A few dimensional arguments would have been so useful here, indeed dimensional quantities are almost completely absent from the book.

The limitations of the book are apparent when the various engine types are discussed. It is clear that the authors have sought, or received, information from only a few specific firms and the resulting selection appears comic to the well-informed British reader.

The chapters on axial and centrifugal compressors are adequate although the latter hardly justifies the theoretical work put into its study, but the Jet Theory section of the chapter on Aircraft Propulsion would confuse any student even if he already knew the subject well. On p. 262 the propulsion efficiency of a rocket as stated must either be regarded as incorrect, or as insufficiently explained. In either case the authors fail to indicate the paradox of rocket efficiencies exceeding 100 per cent.

A useful chapter to the general reader follows in which some of the power generation and transportation turbine units are described, which serves to give some idea of the field of industrial application of gas turbines. There is a relatively long chapter on fuels and combustion that is above the general level of the book, but here again it seems out of place to explain exactly what viscosity is. Surely such elementary physics should be assumed understood. Two chapters on Stresses and Vibration are also quite adequate although the latter is marred by omission of the most important source of compressor blade excitation—blade stalling.

The final chapter on Materials and Metallurgy is not satisfactory. Eleven pages are devoted to tables of properties of commercial alloys which would have been better devoted to explaining what are the real qualities that are required. Obviously high strength at high temperature is required for turbine blades, but this alone cannot be a criterion, whereas strength/density ratio is at least a good starting point. The virtues of a particular alloy can be derived from manufacturers' literature in a few minutes by a well grounded technician, but pages of such literature are no substitute for teaching the correct fundamentals.

This reviewer can find no adequate reason to recommend this book by comparison with others that have already appeared.—H. PEARSON.

THE PRINCIPLES AND PRACTICE OF MANAGEMENT. *Edited by E. F. L. Brech. Longmans Green, 1953. 752 pp. Diagrams. £2 10s. net.*

A comprehensive library of books on Management would, today, contain a large number of volumes bearing titles such as that carried by Mr. Brech's latest work. The vast majority of them, however, would be American and it is therefore an event of some importance when an addition to the literature is made in this country.

There can be no doubt that this new book, containing over seven hundred and fifty pages of text, comes fairly close to its editor's objective of providing a systematic and complete survey of Management principles, with special emphasis on the British point of view.

It will be realised that the task is virtually impossible of achievement unless the field is narrowed severely. Perhaps the main criticism of the book is that this narrowing of the scope has not been enforced rigorously enough by the editor.

To quote a few examples—Work Study is treated in 33 pages, of which about half is devoted to working out various types of standard times in considerable detail. But for a study of *how* to do the job this is far too short and from the point of view of a student of Management it is too long. There are also sections such as "Things a Personnel Manager should know," which gives a list of matters, but neither the answers nor exactly where they are to be learned.

Many teachers of Management subjects will regret the editor's choice of a definition of "Administration" as "the part of Management concerned with the installation and carrying out of the procedures by which the programme is laid down" instead of the "formulation of Policy" which he rejects. This is purely a matter of opinion but there are good arguments for the policy which Mr. Brech opposes.

These few critical remarks must be regarded merely as indications that some disappointment is felt that such an excellent, thorough and comprehensive work displays in some parts a standard lower than that of its major content and the contrast is clear and noticeable.—J. V. CONNOLLY.

STATICALLY INDETERMINATE STRUCTURES. *Professor Chu-Kia Wang. McGraw-Hill, 1953. 424 pp. Diagrams. 60s. net.*

Here is a book obviously intended for the civil engineering fraternity of about degree standard. Nevertheless, it can be used as an excellent reference book by th young aircraft stressman, and will no doubt gladden the heart of many an undergraduate in his final year, whose present problem is both " static " and " indeterminate."

Experience in teaching has shown that while the young engineer can readily understand the principles of the given lecture, the application of these principles is decidedly more difficult and, as stated by the author, " the principles involved are generally simple, but proficiency can be achieved only through practice on a variety of problems."

With this aim in mind, in each chapter the author has given a concise analysis of the underlying principles and has then shown how those principles can be used in a most excellent set of worked examples which Prof. Wang has compiled over a number of years.

The book is divided into two main sections; the first deals by way of an introduction with the solution of statically determinate structures, while the second section, the major portion of the book, outlines the solution of problems which are statically indeterminate.

The first section, the first three chapters, outlines the solution of statically determinate problems by the usual methods of consistent deformation, unit load, and moment area. Each method is clearly developed from first principles and illustrated by the use of worked examples.

The solution of indeterminate beams, rigid frames and trusses in the main section, is done by the method of consistent deformation, the three moment equations, slope deflection and column analogy, while the problems of fixed arches and composite structures are covered adequately.

Criticisms are few. It is thought that some improvement could be made in the introductory section by stating Castigliano's theorem initially, and developing the unit load and moment area methods in the subsequent chapters; while surely, in the chapters dealing with the moment distribution method, it would be of interest to the student to know of Hardy Cross's paper " Analysis of Continuous Frames by Distribution of Fixed End Moments " and of Southwell's work in this field.

These apart, the book is well written, excellently illustrated, and worthy of a place in any library, although the price is rather high.

THERMODYNAMICS AND STATISTICAL MECHANICS. *William P. Allis and Melvin A. Herlin. McGraw-Hill, 1952. 239 pp. Diagrams. 51s. net.*

The authors of this book are members of the physics staff at Massachusetts Institute of Technology, and this is reflected in their essentially physical approach to the subject. The fundamental aspect of the laws of thermodynamics is stressed and the writers develop the subject of statistical mechanics far enough to indicate the reason for these laws. The treatment of the statistical mechanics is from the Boltzmann-Planck point of view, and is based on the classical mechanics of mass-points and permutations among cells in phase space by contrast to the more usual treatment using energy levels and the partition function. The book was written for seniors in physics at M.I.T. as a half-year course which has necessitated, in view of this time limitation and the inclusion of statistical mechanics, the omission of all discussion of chemical equilibria.

The book is sub-divided into two parts: Part A, concerned with heat and thermodynamics, occupies 134 pages, while statistical mechanics, comprising Part B, covers 96 pages. The opening chapter is devoted to a brief consideration of the mathematics required in thermodynamical theory, and it is followed by a chapter on the equation of state in which the authors begin with a short discussion on intermolecular forces and basic types of crystal structure. The next chapter deals with the gas laws and it is particularly well written for the student reader; the last three chapters of Part A are concerned with the first and second laws of thermodynamics.

Part B starts with a chapter on entropy and probability and is followed by the chapter on Manwell-Boltzmann statistics in which paramagnetism and adiabatic demagnetisation receive appropriate consideration. Equipartition of energy, quantum statistics and degenerate gases are the subjects of the last three chapters.

The book is well illustrated with clear line diagrams, and nearly every chapter contains problems suitably chosen to dovetail with the preceding subject matter. The numerical data are quoted in the c.q.s. system, although the reviewer noted one exception (Table 7.1) in which the m.k.s. system was used. Taken as a whole the book should provide an excellent basic text for the honours physics graduate. It is the latest addition to the International Series in Pure and Applied Physics and it fully maintains the high standard of production set for this series. The price is rather high, however, for a student's text book of 230 pages and in a future edition the publishers might consider the possibility of compressing the problems, *e.g.* by using a smaller fount, as in the present volume they occupy over one seventh of the printed space.—R. W. B. STEPHENS.

ADVANCES IN APPLIED MECHANICS, Vol. III. *Edited by R. von Mises and Th. von Kármán. Academic Press Inc., New York, 1953. 324 pp. Diagrams. $9.00 net.*

Contents:—Boundary layer problems in applied mechanics, by G. F. Carrier (19 pp.); The one-dimensional isotropic fluid flow, by O. Zaldastani (39 pp.); Turbulent diffusion: Mean concentration distribution in a flow field of homogeneous turbulence, by F. N. Frenkiel (47 pp.); On aerodynamics of blasts, by H. F. Ludloff (36 pp.); On the presence of shocks in mixed subsonic-supersonic flow patterns, by G. Guderley (40 pp.); Vortex systems in wakes, by L. Rosenhead (11 pp.); Some recent results in the theory of an ideal plastic body, by H. Geiringer (98 pp.); Non-autonomous systems, by A. I. Bellin (26 pp.).

Research in mechanics is making rapid strides and the literature of the subject has increased enormously in size since the 1939-45 War. Indeed, whole new fields have been opened up and now have their specialists. There are few people who have the time and the inclination to try to keep abreast of developments by reading the great flood of papers published in journals or issued as laboratory or government reports. Most people adopt the sensible plan of confining their efforts to keep well-informed to the reading of review articles published a few years after the advances have been made. There are not many media suitable for such surveys, midway between a paper and a monograph in length, and the initiation of *Advances in Applied Mechanics* some years ago was therefore a very welcome and useful step. (But surely the editors have in mind *theoretical* mechanics?)

This is the third volume of the series to appear, and it contains some interesting articles. To single out any of them for praise or criticism seems to be invidious, since

there were some I could read with ease and familiarity, some only with hesitant appreciation and understanding, and some which I did not read at all. But I think it is fair to say, from a reasonably complete coverage, that the quality of the articles is very uneven. Only by stretching the meaning of the word can a couple of them be said to describe advances. Some others are very good and have just the degree of generality that the non-specialist reader hopes to find. I have the feeling that the opportunities offered by the original conception of the *Advances in Applied Mechanics* have not been used quite as well as they might have been. Clearly it all depends on the choice of the authors. I think there is room for more authors who, among other qualities, rely on talk about the meaning of what they are doing rather than on mathematics.—G. K. BATCHELOR.

COMMUNICATION THEORY. *Edited by Willis Jackson. Butterworths Scientific Publications, London 1953. 532 pp. Diagrams. 65s. net.*

This book is a collection of some forty specialist papers read at a four day Symposium on " Applications of Communication Theory," together with the relevant discussions which took place. It is edited by Willis Jackson who at that time was Professor of Electrical Engineering at Imperial College, London, and his opening address at the Symposium forms the introduction to this book.

The main aim of the papers is the consideration of the practical aspect of the now comprehensive theory of the communication of information to the fairly broad field of electrical communication. The papers are grouped under six main topics but there is an introductory paper by Dr. Gabor summarising the present position of the theory of communication. It is not, nor was it intended to be, an elementary introduction to the theory but is rather in the nature of an integrating paper forming a theoretical background which could then be regarded as common knowledge by the various contributors to avoid needless repetition.

The first section dealing with transmission systems and methods of coding, forms about a quarter of the book and contains nine papers. The second section brings us to the half-way point and deals with the presence of noise in transmission systems. This also contains nine papers. Much of the work is of a practical nature such as, a description of an analogue computer for determining correlation functions, methods of video signal integration using acoustic delay lines, magnetic and electrastatic storage systems and a special cathode ray storage tube. The theoretical side is by no means neglected and this section is probably the most important in the book because, largely, the practical work is a direct outcome of theoretical application.

Other sections are:—Characteristics of Transmission Channels, Application to Television, Hearing, and finally, Transmission and Analysis of Speech. There are four papers at the end of the book on closely related topics; information generators, statistical study of language, semantic information, and an application of information theory to the optical problem of the resolving power of microscopes.

All the papers are of an exceedingly high standard; some of the theoretical works, as shown by the lively discussions, are on debatable topics and are not in the final form. Thus the book is of most use to the specialist in this, or closely related, fields of work. It is to be recommended to all who wish to become specialists in the application of communication theory.—JOHN C. WEST (The University, Manchester).

GASDYNAMIK. *K. Oswatitsch. 456 pp. 300 diagrams. 3 charts. Springer, Vienna, 1952. £6 13s.*

The conventionally brought up aerodynamicist may be as happy as this reviewer to know of a work which will give him the feel of gas dynamics. The subject has wide terms of reference. The only branch of the physics of fluid motion that is not included appears to be that for which the term incompressible is apposite. In this volume we find ourselves restricted to considerations of—on the whole—inviscid continuous gas which is not subject to body force. Furthermore, no great play is made of acoustics. We are told in the author's introduction that so much has been published on what remains that it is scarcely possible to read it all. Yet he has managed to give us more than 350 references, some of which only appeared in 1952. In the reviewer's opinion this is a minor blemish in an excellent textbook—one of the very few he could find!—since such a plethora of reference works may lead the uncomprehending student into temptation to put off the effort of learning some difficult point.

This book is neither for the undergraduate nor for the expert in any of the subdivisions of the subject. Therein lies its value for the young post-graduate and like intellects who are attracted to it. Dr. Oswatitsch is a master in this field. He has an unusually good physical insight into the mechanics of fluid motion and provides plausible explanations for most of the phenomena which he describes. He does not shirk the use of mathematics to put the matter on a quantitative basis wherever possible. The rigour of the treatment has slowed up the progress of the reviewer here and there but the work can be understood without recourse to the mathematician.

The necessary elementary thermodynamics is covered in the first chapter. The second and third chapters hold a very thorough and occasionally novel treatment of one-dimensional steady and unsteady flow. General equations and theorems are then negotiated with ease and elegance. This will also apply to those readers who are ready to bring a little knowledge of vectors to the task. Then follows an awkwardly placed chapter concerned with the application of the previous—largely to idealised propulsion problems. Next we meet the velocity potential and the stream function for the plane and for axi-symmetrical flow, the change in type of the linearised equation of potential as $M \gtrless 1$, sources and vortices, Prandtl-Meyer flow, some classical transformations and similarity laws. There is an admirably careful discussion of boundary conditions in this chapter. Then there follow chapters which are headed: steady plane and axi-symmetrical subsonic flow, the like for supersonic flow, steady transonic flow, some particular steady and unsteady three-dimensional flows,—including swept plan forms, deltas, Evvard and other tip theories and so on. These six chapters occupy 232 closely reasoned pages and form a valuable connected account of high-speed aerodynamics at post-graduate level. Characteristics techniques are used rather more than in conventional treatments and conical flow problems are not solved by explicit use of conformal transformation. These are author's preferences. The penultimate chapter contains a brief discussion of viscous compressible flow problems. It is clearly not intended to be as thorough as the earlier part of the book. This also applies to the chapter on experimental methods with which the work is concluded.

Only a few private individuals will be able to afford to buy this excellent and expensive work; technological libraries cannot afford to be without it!—P.T.F.

THE GREEN AND RED PLANET. *Hubertus Strughold. University of New Mexico Press 1953. 107 pp. 17 Figures. 4 dollars net.*

This interesting little book examines through the eyes of a physiologist the possibility that life as we know it may exist on Mars or on other planets. To make this examination the author draws from a wide field of knowledge of the biological sciences, astronomy and of the earth's atmosphere.

The environment that Martian inhabitants would encounter is examined with regard to temperature, pressure and composition of atmosphere and radiation flux, and an estimate is made of what forms of life would survive under these conditions. As the author specifically excludes from consideration life based on chemical processes outside our experience the book becomes a relatively straightforward matching of the known or estimated physical conditions against the limits of survival of various forms of animal and plant life, as have been demonstrated by experiment here on earth.

In the section contrasting the Martian atmosphere with that of the Earth a useful summary of the effects of anoxia on man at various altitudes is provided. To workers in aviation this may well be the most useful section of the book and indeed the most interesting, for here the author is on sure ground and dealing with a subject where speculation is not required.

Otherwise the book is useful primarily as a digest of a wide field of science. Its principle virtues will be stimulation of interest to the layman and a convenient summary of data to the worker in aviation or space medicine.—H. L. ROXBURGH (Institute of Aviation Medicine).

THE OBSERVER'S BOOK OF AIRCRAFT. *William Green and Gerald Pollinger. Frederick Warne, London 1953. 287 pp. Illustrated. 5s. net.*

The second edition of the Observer's Book of Aircraft contains 171 descriptions and 284 illustrations of modern aircraft. For small boys and others who cannot afford "Janes," this little volume is an invaluable addition to the "recognisers" library. A photograph and three elevation silhouettes are given of all the best known aircraft and the prefatory pages explain the books arrangement, define Mach number and list international Civil aircraft markings. No house over which aircraft pass should be without a copy.

EAGLE BOOK OF AIRCRAFT. *J. W. R. Taylor. Hulton Press Ltd., London 1953. 192 pp. Illustrated. 10s. 6d. net.*

There is really no point in reviewing this book.

There can be very few boys (7 plus) who do not see the *Eagle* weekly. There can be very few boys (same age group) who have not an absorbing interest in aeroplanes.

The "Eagle Book of Aircraft" is, therefore, self-recommended for boys of all ages from 7 to 70.

The illustrations are unusual and beautifully clear and there are many of those excellent cutaways which are so well known to those caddish fathers who grab their sons' copy on Thursday mornings.

The text covers history, technicalities and modern developments in a most readable way. Altogether one of the best and most intelligent books on aeronautics that cater for youth that has appeared.

THERMODYNAMICS APPLIED TO HEAT ENGINES. *E. H. Lewitt. Fifth Edition.* 820 *pp.* 333 *figures. Pitman, London,* 1953. 30*s. net.*

This is a new edition of a book first published in 1933; the previous edition was reviewed in the JOURNAL of August 1950. The book has been brought up to date by the addition of new work to the chapter on gas turbines and jet propulsion, dealing with thrust augmentation for jet propulsion and on supersonic ram-jets. A new chapter has also been added containing work on humidity of air, the heat pump, liquefaction of gases by the Linde and the Claude cycles, and turbo-expanders; a temperature-entropy chart for low temperature air has been included as a folding inset. An article on entropy is also included in which the author has developed a conception of entropy from the availability of heat for conversion into work.

COVER OF DARKNESS. *Roderick Chisholm. Chatto & Windus, 1953. 222 pp. Illustrated. 12s. 6d. net.*

Written by an airman very readily able to illuminate with apt phrase his perception of flying and its reaction on mind and senses, this most interesting book describes his experiences in the evolution of night-fighter technique. In 1940, when as a Flying Officer he rejoined the R.A.F., Air Commodore Chisholm had to relearn the art of flying, and particularly night flying. He found it hard to win self-confidence, yet his dispassionate power of self-analysis makes the observations on this phase of flight training particularly noteworthy and valuable. Facilities have greatly altered and improved since then; nevertheless, the young pilot of to-day has to find personal victory in much the same process of psychological conditioning.

Although of particular interest to pilots, and certainly of nostalgic recollection to those who fought at night, *Cover of Darkness* offers equally to those who do not fly a fascinating insight of a particularly gallant and remarkable chapter of the war. From ineffective beginnings, when radar was only partially reliable and the methods of night interception still undeveloped, the author takes the reader with him on night combats, and thence to the arduous yet successful development of the techniques which found their ultimate perfection in the Mosquito and those who so courageously flew this remarkable aeroplane in the long struggle under cover of darkness.

Later in the war, after commanding an operational night fighter development unit, Air Commodore Chisholm's great experience of night flying was applied to the formation of a Radio Counter Measures Group where the purpose was to aid our bomber crews, then flying on the huge thousand bomber raids, by confusing the enemy with deceptive interference and exploitation of radio waves. In turn this led to jamming, and "spoof" operations, and offensives against enemy night fighters.

Of the author and his companions, Air Chief Marshal Sir Will Elliott writes in a brief Foreword which epitomises the book: "They knew and 'prized the doubt.' They understood what was at stake, and they played the game with skill and care, the patience and endurance, which alone brings to great adventurers great rewards."—H. J. PENROSE.

FATIGUE AND FRACTURE OF METALS. *A Symposium held at the Massachusetts Institute of Technology, June 1950. Edited by Murray and Hunsaker. 313 pp. Illustrated. Chapman and Hall, 1952. 48s.*

1. General Survey of the Problem of Fatigue and Fracture. Gensamer. 2. The Fatigue Problem in Airplane Structures. Dryden. Rhode and Kuhn. 3. Brittle Fracture and Fatigue in Ships. Jonassen. 4. Brittle Fracture and Fatigue in Machinery. Peterson. 5. Internal Stresses and Fatigue. Horger and Neifert. 6. Designing for Fatigue. Templin. 7 Fundamentals of Brittle Behavior in Metals. Orowan. 8. Experimental Study on Temper Brittleness of Slightly Alloyed Carbon Steel. Jacquet and Weill. 9. The Statistical Aspect of Fatigue Failures and its Consequences. Weibull. 10. A Review of Cumulative Damage in Fatigue. Newmark. 11. Significance of Transition Temperature in Fatigue. MacGregor. 12. The Influence of Metallographic Structure on Fatigue. Teed. 13. Fatigue at Elevated Temperatures. Grant. 14. The Techniques of Physical Metallurgy for Studying Fatigue Damage. Norton.

It seems a pity that the discussions of the papers have not been included, because these often help to clarify points which in a written paper may be obscure.

The reader who expects to get a general picture of the fatigue problem from the first paper will be disappointed. Except for a brief reference to fatigue at the beginning, the paper deals entirely with a discussion of transition temperatures in steel plates. On this subject, however, the paper is informative and well-written, and the author at the outset makes a plea for clearing up confusion in the use of the various terms and definitions employed. This, regrettably, is not the case throughout the book: in places, for example, one has to pause and wonder whether the words "tensile stress" are intended to mean a stress in the nature of a pull or the ultimate tensile strength of the material.

The problem of transition temperature and its significance is dealt with very admirably and in great length in Paper 11, and it forms quite a large part of the subject matter of one or two other papers.

The second paper lives up to its title rather better. It gives quite a good *general* survey of the problem that faces the aircraft designer, and contains some useful data on gust measurements. Unfortunately, there is nothing in it to help in the matter of *detail* design.

The subject of cumulative damage has now been on trial for a long time, and has been found wanting. It is refreshing therefore to have a paper on such a controversial subject written in an entirely different way. The simple cumulative damage rule, as originally proposed by Miner and others, gives $\Sigma(n/N)=1$, but numerous experiments to test this rule give values of $\Sigma(n/N)$ of anything from 0·02 to about 30, depending on the conditions of test. Bq adopting such devices as using a minimum $S-N$ curve and suitably choosing the various stress levels and the order in which they are imposed, the limits of $\Sigma(n/N)$ can, it is true, be brought much nearer together. But in view of the considerable amount of scatter generally, it is rather surprising to read that normally we may accept the rule as given earlier, while in unusual cases we might take $\Sigma(n/N)=0{\cdot}8$. This seems altogether too sweepingly arbitrary.

The phenomenon of notch-brittleness of materials which are basically ductile is intimately associated with the problem of fatigue. A clear exposition of this subject is given in Paper 7. The way in which cracks or notches induce a state of tri-axial tensions and the effect of this on notch-brittleness are discussed.

The paper dealing with the statistical aspects of fatigue is a very convincing one, and it certainly provides an excellent résumé of the subject. The stress S and number of cycles N are plotted on a log log/log log basis, resulting, for the materials tested, in a straight line, so that the relationship between S and N can be expressed in a simple analytical manner. To obtain the $S-N$ band, the same number of specimens are tested at each level of stress, a device intended to translate the considerable scatter obtained in endurance, into degrees of probability.

Paper 12, dealing with the metallographic side of the subject, provides a very comprehensive treatment in a relatively short space and includes data on aluminium and magnesium alloys as well as on steels. It is divided into five parts: the first is an Introduction, and the other four parts deal respectively with the effect of grain size, precipitation hardening, inclusions, and residual stresses.

Paper 6, on Designing for Fatigue, contains a small amout of data relating to fatigue tests on aluminium alloy riveted joints and aluminium alloy box beams.

The last paper, dealing with the technique for studying fatigue damage, has one main fault—it is much too short. No mention is made, incidentally, of the electron microscope, or of the possibilities of using photo-elasticity.

The remaining papers are in general well-written, and contain useful data, though confined mostly to steels.

The book as a whole is exceedingly well produced. There is no question of its value to anyone in any way concerned with the problem of fatigue. There are a few printing errors. On page 82, in the formula for notch sensitivity given in Fig. 12, K_f in the denominator should be K_t. On page 186, Table I, the first figure in the last column should be $-0{\cdot}002$, not $-0{\cdot}02$. On page 193, Table III, column 3, the last figure but one in the left-hand column of figures should be 3·88. On pages 189 and 190, the curves in Figs. 2 and 3 have become inadvertently mixed. The curve in Fig. 2 is the $S-N$ curve for $P=0{\cdot}1$, not for $P=0{\cdot}5$, and that in Fig. 3 is for $P=0{\cdot}5$, not $P=0{\cdot}1$.

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Reports

AERODYNAMICS

Boundary Layer

See also Wings and Aerofoils

Determination of boundary-layer transition Reynolds numbers by surface-temperature measurement of a 10° cone in various N.A.C.A. supersonic wind tunnels. A. O. Ross. N.A.C.A. Technical Note 3020 (*October* 1953).

An investigation was conducted in the Ames and Lewis supersonic tunnels to determine transition Reynolds numbers on a 10° cone at zero angle of attack. Measurements of surface temperature at known increments along the cone were made for the case of negligible heat transfer and negligible wind-tunnel pressure gradients.—(1.1.2).

The compressible laminar boundary layer with heat transfer and small pressure gradient. G. M. Low. N.A.C.A. Technical Note 3028 (*October* 1953).

A perturbation method for the calculation of velocity and temperature profiles, skin-friction and heat transfer characteristics for two-dimensional compressible laminar boundary layers with heat transfer and small arbitrary gradient is presented. The permissible pressure gradients include those of a form and magnitude usually encountered over slender aerodynamic shapes in supersonic flight.—(1.1.1 × 1.9.1).

Preliminary experimental investigation of low-speed turbulent boundary layers in adverse pressure gradients. V. A. Sandborn. N.A.C.A. Technical Note 3031 (*October* 1953).

Measurements of velocity profiles and skin friction in subsonic turbulent boundary layers with adverse pressure gradients are presented. The skin friction was measured by using a local heat-transfer instrument and employing a calibrated relation between heat transfer and skin friction.—(1.1.3).

Compressible Flow

See also Wings and Aerofoils

An experimental investigation of transonic flow past two-dimensional wedge and circular-arc sections using a Mach-Zehnder interferometer. A. E. Bryson, Jr. N.A.C.A. Report 1094 (1952).

Interferometer measurements are given of the flow fields near two-dimensional wedge and circular-arc sections at zero angle of attack. Pressure distributions and drag coefficients as functions of Mach number were obtained and the wedge data are compared with theory.—(1.2.2 × 1.10.2.1).

Experimental investigation of base pressure on blunt-trailing-edge wings at supersonic velocities. D. R. Chapman, W. R. Wimbrow and R. H. Kester. N.A.C.A. Report 1109 (1952).

The pressures acting on the base of blunt-trailing-edge aerofoils have been measured at Mach numbers of 1·25, 1·5, 2·0 and 3·1, and at Reynolds numbers from 0·2 to 3·8 million. Data are presented for 29 profiles both with laminar and with turbulent boundary layers approaching the trailing edges of the wings. The base pressure is found to be a function primarily of Mach number and the ratio of the boundary layer thickness at the trailing edge to the trailing-edge thickness.—(1.2.3 × 1.10.2.1).

Internal Flow

Concerning the flow about ring-shaped cowlings. Part XII—Two new classes of circular cowls. D. Küchemann and Johanna Weber. N.A.C.A. Technical Memorandum 1360 (*October* 1953).

For application in practice for annular radiator fairings and similar arrangements, two new classes of circular cowls are developed by theoretical method, and investigated in a systematic test series regarding their behaviour under various working conditions.—(1.5).

Concerning the flow on ring-shaped cowlings. Part XIII—The influence of a projecting hub. D. Kuchemann. N.A.C.A. Technical Memorandum 1361 (*October* 1953).

The influence of thickness and length of a hub projecting from an inlet opening was investigated on one of the two new classes of circular cowls reported in N.A.C.A. T.M. 1360.—(1.5).

A note on secondary flow in rotating radial channels. J. J. Kramer and J. D. Stanitz. N.A.C.A. Technical Note 3013 (*October* 1953).

A general vector differential equation for the vorticity component parallel to a streamline is derived for steady, non-viscous and incompressible flow in a rotating system. This equation is then simplified by restricting it to rotating radial channels and by making further simplifying assumptions. This simplified equation is used to solve for the secondary vorticity, the vorticity component parallel to the streamline, in three special cases involving different stream-tube geometries.—(1.5).

An experimental investigation of secondary flow in an accelerating, rectangular elbow with 90° of turning. J. D. Stanitz, W. M. Osborn and J. Mizisin. N.A.C.A. Technical Note 3015 (*October* 1953).

Secondary flow tests were conducted on an accelerating elbow with 90° of turning designed for prescribed velocities that eliminate boundary-layer separation by avoiding local decelerations along the walls. Secondary flows were investigated for six boundary-layer thicknesses generated on the plane walls of the elbow by spoilers upstream of the elbow inlet.—(1.5.1).

Performance Estimation

Tests on a Whirlwind aircraft in the Royal Aircraft Establishment 24-ft. wind tunnel (in two parts). T. V. Somerville, and R. R. Duddy and G. H. L. Buxton. R. & M. 2603 (*June* 1940, *published* 1953).

Tests were made in the 24-ft. wind tunnel during March and April 1940 on the Whirlwind aircraft to find if simple modifications can be introduced which will decrease its drag. The drag analysis is not complete and is focused chiefly on the drag due to leaks, cooling and excrescences. A complete record of the tests together with explanatory paragraphs is given in the tables of this note.—(1.7 × 4.2).

Stability and Control

See also Design and Construction

A general treatment of static longitudinal stability with propellers, with application to single-engined aircraft. E. Priestley. R. & M. 2732 (*May* 1944, *published* 1953).

A general method of treatment of stick-fixed static longitudinal stability with propellers is given, distortion and compressibility effects being neglected. Model full-throttle data on some single-engined fighters are analysed for the flaps-up condition to establish a basis of estimation of effect of propeller on stability for this type of design. The general effect of propellers on manœuvre point, more particularly the effect on H_m-K_n, is considered in an appendix.—(1.8.2).

A theoretical analysis of the effects of fuel motion on airplane dynamics. A. A. Schy. N.A.C.A. Report 1080 (1952).

The general equations of motion for an aeroplane with a number of spherical fuel tanks are derived. These equations are applied to two cases with two fuel tanks located in the plane of symmetry. The same type of analysis may be applied to arbitrarily shaped tanks; therefore, the most general conclusions as to the effects of the fuel motion on aeroplane dynamics also apply for arbitrarily shaped tanks. Formerly T.N. 2280.—(1.8.1 × 1.8.2).

Flight investigation of a mechanical feel device in an irreversible elevator control system of a large airplane. B. P. Brown, R. G. Chilton and J. B. Whitten. N.A.C.A. Report 1101 (1952).

NOTE:—*The figures in parentheses at the end of each Summary are for office use only.*

Data are presented showing the flight characteristics of a large aeroplane having a control-surface booster and mechanical feel device in the elevator-control system. The tests were made with various force gradients provided by the adjustable feel device. The booster was set to operate at a very high boost ratio throughout the tests so that the measured or apparent stick-free stability would be influenced only slightly by the aerodynamic hinge moments.—(1.8.2).

A method of deriving frequency-response data for motion of the center of gravity from data measured on an aircraft at locations other than the center of gravity. J. M. Eggleston. N.A.C.A. Technical Note 3021 (*October* 1953).

A method is given for deriving time-response and frequency-response data for angle of attack and normal acceleration of the centre of gravity of an aircraft when these data are measured at locations on the aircraft other than the centre of gravity and when the pitching velocity is not recorded.—(1.8).

THERMO-AERODYNAMICS

See also BOUNDARY LAYER

An analysis of laminar free-convection flow and heat transfer about a flat plate parallel to the direction of the generating body force. S. Ostrach. N.A.C.A. Report 1111 (1953).

A formal and general analysis of the free-convection flow about a flat plate parallel to the direction of the generating body force is made, and velocity and temperature distributions for Prandtl numbers of 0·01, 0·72, 0·733, 1, 2, 10, 100, 1,000, and large Grashof numbers are computed. It is shown that velocities and Nusselt numbers of the same order of magnitude as those associated with forced-convection flows can be obtained under free-convection conditions. A flow and a heat-transfer parameter are derived from which the important physical quantities can be computed.—(1.9.1).

Heat transfer and skin friction by an integral method in the compressible laminar boundary layer with a streamwise pressure gradient. I. E. Beckwith. N.A.C.A. Technical Note 3005 (*September* 1953).

A simplified method has been developed for the calculation of heat transfer and skin friction in the compressible laminar boundary layer with an arbitrary Prandtl number near unity and an arbitrary streamwise pressure gradient and wall temperature distribution. The method has been extended to the calculation of heat transfer under conditions of equilibrium dissociation in the boundary layer. By means of a suitable transformation the method is also applied to a body of revolution with a boundary-layer thickness of the order of the body radius.—(1.9.1).

Comparison of effectiveness of convection-, transpiration-, and film-cooling methods with air as coolant. E. R. G. Eckert and J. N. B. Livingood. N.A.C.A. Technical Note 3010 (*October* 1953).

Material necessary for a comparison of the cooling requirements of film and transpiration cooling with conventional convection cooling is presented. Both laminar and turbulent flow, with and without radiation, are considered for a flat plate.—(1.9.1).

Analysis of turbulent heat transfer and flow in the entrance regions of smooth passages. R. G. Deissler. N.A.C.A. Technical Note 3016 (*October* 1953).

A previous analysis for fully developed turbulent heat transfer and flow with variable fluid properties is extended and applied to the entrance regions of smooth tubes and parallel flat plates. Integral heat-transfer and momentum equations are used for calculating the thicknesses of the thermal and flow boundary layers. The effect of variable properties is determined for the case of uniform heat flux, uniform initial temperature distribution, and fully developed velocity distribution. A number of other cases in which the fluid properties are constant are analysed.—(1.9.1).

WINGS AND AEROFOILS

See also COMPRESSIBLE FLOW

Tests on a swept-back wing and body in the compressed air tunnel. C. Salter, C. J .W. Miles and Miss H. M. Lee. R. & M. 2738 (*May* 1950, *published* 1953).

The model consisted of a swept-back wing of symmetrical section and a long cylindrical body. The aspect ratio was 3·29, the taper ratio 2·75, the sweep of the quarter-chord line 42·5 deg., the maximum thickness/chord ratio at the root 8·6 per cent. and at the tip 10 per cent. It has been tested over a range of Reynolds numbers of $0{\cdot}5 \times 10^6$ to 13×10^6 and results are given of lift drag and pitching moment for angles of incidence up to 30 deg.—(1.10.2.2).

Two- and three-dimensional unsteady lift problems in high-speed flight. H. Lomax, M. A. Heaslet, F. B. Fuller and Loma Sluder. N.A.C.A. Report 1077 (1952).

The problem of transient lift on two- and three-dimensional wings flying at high speeds is discussed as a boundary-value problem for the classical wave equation. Kirchhoff's formula is applied so that the analysis is reduced, just as in the steady state, to an investigation of sources and doublets. The applications include the evaluation of indicial lift and pitching-moment curves for two-dimensional sinking and pitching wings flying at Mach numbers equal to 0, 0·8, 1·0, 1·2, and 2·0.—(1.10.1.2).

Method for calculating lift distributions for unswept wings with flaps or ailerons by use of nonlinear section lift data. J. C. Sivells and Gertrude C. Westrick. N.A.C.A. Report 1090 (1952).

A method is presented for calculating lift distributions for unswept wings with flaps or ailerons using non-linear section lift data. This method is based upon lifting-line theory and is an extension to the method described in N.A.C.A. Rep. 865. Simplified computing forms containing detailed examples are given for both symmetrical and asymmetrical lift distributions. A few comparisons of experimental and calculated characteristics are also presented.—(1.10.1.2 × 1.3).

Chordwise and compressibility corrections to slender-wing theory. H. Lomax and Loma Sluder. N.A.C.A. Report 1105 (1952).

Corrections to slender-wing theory are obtained by assuming a spanwise distribution of loading and determining the chordwise variation which satisfies the appropriate integral equation. Such integral equations are set up in terms of the given vertical induced velocity on the centre line or, depending on the type of wing plan form, its average value across the span at a given chord station. The chordwise distribution is then obtained by solving these integral equations. Results are shown for flat-plate, rectangular, and triangular wings.—(1.10.1.2).

Experimental aerodynamic derivatives of a sinusoidally oscillating airfoil in two-dimensional flow. R. L. Halfman. N.A.C.A. Report 1108 (1952).

Experimental measurements of the aerodynamic reactions on a symmetrical aerofoil oscillating harmonically in a two-dimensional flow are presented and analysed. Harmonic motions include pure pitch and pure translation, for several amplitudes and superimposed on an initial angle of attack, as well as combined pitch and translation. Considerable consistent data for pure pitch were obtained from a search of available reference material, and several definite Reynolds number effects are evident.—(1.10.1.1).

The aerodynamic characteristics of an aspect-ratio-20 wing having thick airfoil sections and employing boundary-layer control by suction. Bennie W. Cocke, Jr., Marvin P. Fink and Stanley M. Gottlieb. N.A.C.A. T.N. 2980. (*August* 1953.)

An investigation has been conducted to study the aerodynamic characteristics of an aspect ratio-20 wing employing thick aerofoil sections and boundary layer control by suction. Data from models tested in the Langley full-scale tunnel and the Langley low-turbulence pressure tunnel are included.—(1.10.2.2 × 1.1.5).

Supersonic flow past oscillating airfoils including nonlinear thickness effects. Milton D. Van Dyke. N.A.C.A. T.N. 2982. (*July* 1953.)

A solution to second order in thickness is derived for harmonically oscillating two-dimensional aerofoils in supersonic flow. For slow oscillations of an arbitrary profile, the

result is found as a series including the third power of frequency. For arbitrary frequencies, the method of solution for any specific profile is indicated, and the explicit solution derived for a single wedge.—(1.10.1).

The effects of camber on the variation with Mach number of the aerodynamic characteristics of a 10-percent-chord modified N.A.C.A. four-digit-series airfoil section. Albert D. Hemenover. N.A.C.A. T.N. 2998. (*September* 1953.)

The effects of camber on the variation with Mach number of the aerodynamic characteristics of a 10-percent-chord-thick modified N.A.C.A. four-digit-series aerofoil section are determined from tests in the Ames 1- by 3-1/2-foot high-speed wind tunnel at Mach numbers from 0·3 to 0·9 and corresponding Reynolds numbers from 1×10^6 to 2×10^6.—(1.10.2.1).

Effects of finite span on the section characteristics of two 45° *sweptback wings of aspect ratio* 6. *Lynn W. Hunton. N.A.C.A. T.N.* 3008. (*September* 1953.)

A study has been made of the finite-span effects on the local loading characteristics of two swept-back wings at low speed with a view toward providing some insight into the usefulness of two-dimensional section data and span-loading theory for determining the section characteristics of a swept wing. The analysis is based on comparisons of local pressure distributions and local lift characteristics on the wings with comparable two-dimensional section data, all of which were available at large scale.—(1.10.2.2).

Velocity potential and air forces associated with a triangular wing in supersonic flow, with subsonic leading edges, and deforming harmonically according to a general quadratic equation. C. E. Watkins and J. H. Berman. N.A.C.A. Technical Note 3009 (*September* 1953).

The velocity potential for a triangular wing with subsonic leading edges experiencing harmonic deformations in supersonic flow is treated herein. The oscillations considered are such that the amplitude of distortion of the wing can be represented by a general quadratic equation. The velocity potential is derived in the form of a power series in terms of the frequency of oscillation. The material constitutes an extension of the work given in N.A.C.A. Report 1099.—(1.10.1.2).

Calculated spanwise lift distributions and aerodynamic influence coefficients for unswept wings in subsonic flow. F. W. Diederich and M. Zlotnick. N.A.C.A. Technical Note 3014 (*September* 1953).

Spanwise lift distributions have been calculated for nineteen unswept wings with various aspect ratios and taper ratios and with a variety of angle-of-attack or twist distributions, including flap and aileron deflections, by means of the Weissinger method with eight control points on the semi-span. Also calculated were aerodynamic influence coefficients which pertain to a certain definite set of stations along the span; several methods are presented for calculating aerodynamic influence coefficients for stations other than those stipulated.—(1.10.1.2).

The flow about a section of a finite-aspect-ratio N.A.C.A. 0015 *airfoil on a transonic bump. J. A. Mellenthin. N.A.C.A. Technical Note* 3036 (*October* 1953).

Pressure distributions were measured at one spanwise station on a semi-span rectangular wing model having an N.A.C.A. 0015 aerofoil section and a moderate aspect ratio. The tests were conducted on a transonic bump at Mach numbers from 0·4 to 1·06. Pressure-distribution plots were integrated to obtain the section lift, drag, and pitching-moment coefficients.—(1.10.2.1 × 1.2.2).

Helicopter Aerodynamics

The induced velocity field of a rotor. K. W. Mangler and H. B. Squire. R. & M. 2642 (*May* 1950, *published* 1953).

A short account and the results of a theoretical investigation of the velocity field induced by a lifting rotor is given. The computation is based on the assumptions that the rotor is lightly loaded and that it has an infinite number of blades. This is applied to calculate the induced velocity distribution for disc incidence of 0, 15, 30, 45 and 90 deg. For the downwash at the rotor itself (the normal component of the induced velocity) the Fourier coefficients are given, as they are needed for the calculation of the blade motion.—(1.11).

Method for studying helicopter longitudinal maneuver stability. K. B. Amer. N.A.C.A. Technical Note 3022 (*October* 1953).

A theoretical analysis of helicopter manoeuvre stability is made and the results are compared with experimental results for both a single- and a tandem-rotor helicopter. Techniques are described for measuring in flight the significant stability derivatives for use with the theory to aid in design studies of means for achieving marginal manoeuvre stability for a prototype helicopter.—(1.11).

Testing and Instruments

See also Electronics

Note on the development of a trailing-edge flap as a stall detector. C. F. Bethwaite. College of Aeronautics Report 75. (*September* 1953.)

A trailing-edge flap protruding slightly outside the wing profile has been developed as a stall detector and flight trials on Anson and Provost aircraft have proved successful. The device gave an adequate stall warning for all conditions of power and landing flaps and operated satisfactorily before stalls from steep turns.—(1.12 × 5.3).

Tables of differential pressures corresponding to the readings of air speed indicators. F. W. Hooton. A.R.L. Australia. Flight Note 19. (*August* 1953.)

Tables are presented which give the differential pressures (in lb./ft.^{2}) corresponding to the readings (in miles per hour and knots) of air speed indicators calibrated according to the British convention, for increments of one m.p.h. and one knot respectively up to the speed of sound at sea level. Conversion factors to other units of pressure are included.—(1.12).

New experiments on impact-pressure interpretation in supersonic and subsonic rarefied air streams. F. S. Sherman. N.A.C.A. T.N. 2995. (*September* 1953.)

Results are presented of an experimental investigation of impact-pressure interpretation in supersonic and subsonic rarefied air streams at Mach numbers from 0·1 to 0·7 and 1·7 to 3·4 and in the Reynolds number range from 2 to 800.—(1.12).

*Studies of the use of Freon-*12 *as a wind-tunnel testing medium. Alberet E. von Doenhoff, Albert L. Braslow and Milton A. Schwartzberg. N.A.C.A. T.N.* 3000. (*August* 1953.)

A number of studies relating to the use of Freon-12 as a substitute medium for air in aerodynamic testing have been made. The use of Freon-12 instead of air makes possible large savings in wind-tunnel drive power. Methods for predicting aerodynamic characteristics of bodies in air from data obtained in Freon-12 have been developed, which provide substantial agreement in all cases for which comparative data are available.—(1.12).

AEROELASTICITY

See also Structures—Loads

Measurements of the aerodynamic derivatives for a horn-balanced elevator. N. C. Lambourne, A. Chinneck and D. B. Betts. R. & M. 2653 (*January* 1949, *published* 1953).

The results of measurements by a forced oscillation method of the direct derivatives (aerodynamic stiffness and damping) for a horn-balanced elevator are given. The tests were made at low air speeds on a complete wing-fuselage-tail model at 0 deg. and 10 deg. incidence in a wind tunnel. Some information was obtained on the effect of mean elevator angle on the derivatives when the model was at the high incidence. Measurements were also made with trailing-edge cords and transition wires in position.—(2 × 1.3.2).

An iterative transformation procedure for numerical solution of flutter and similar characteristic-value problems. M. L. Gossard. N.A.C.A. Report 1073. (1952.)

The idea of the iterative transformation procedure suggested by H. Wielandt is explained. Application of the procedure

to ordinary natural-vibration problems and to flutter problems is shown in numerical examples. Comparisons of computed results with experimental values and with results obtained by other methods of analysis are made. This report was formerly N.A.C.A. Technical Note 2346.—(2 × 33.1.2).

DESIGN AND CONSTRUCTION

See also AERODYNAMICS—PERFORMANCE ESTIMATION

A preliminary study of the problem of designing high-speed airplanes with satisfactory inherent damping of the Dutch roll oscillation. J. P. Campbell and Miss M. O. McKinney, Jr. N.A.C.A. Technical Note 3035 (*October* 1953).

Some preliminary results are given of a theoretical investigation to determine what design features appear most important in designing modern high-performance aeroplanes to have the greatest possible inherent stability of the Dutch roll oscillation in order that the need for artificial stabilising devices can be minimised. These preliminary results cover fighter aeroplanes at subsonic speeds.—(4.2.5 × 1.8.1).

AIRCRAFT OPERATION

See also AERODYNAMICS—TESTING AND INSTRUMENTS

Experiments on rigid parachute models. R. Jones. R. & M. No. 2520. (*November* 1943, *published* 1953.)

The investigations described in this report were undertaken for the Aeronautical Research Committee to study the effect of porosity on the stability of a parachute. Experiments were made to obtain data which could be applied in the usual way to the standard equations of motion. These data, drag, lateral force and yawing moment, can only be obtained on rigid models, and it was suggested that porosity might be simulated in a rigid model by perforating the canopy. The "porous" model had holes drilled approximately $\frac{1}{4}$ in. apart along parallels also $\frac{1}{4}$ in. apart (measured along the arc of the generator).—(5.3).

Accelerations and passenger harness loads measured in full-scale light-airplane crashes. A. Martin Eiband, Scott H. Simpkinson and Dugald O. Black. N.A.C.A. T.N. 2991. (*August* 1953.)

Full-scale light-aeroplane crashes simulating stall-spin accidents were conducted to determine the decelerations to which occupants are exposed and the resulting harness forces encountered in this type of accident. Crashes at impact speeds from 42 to 60 m.p.h. were studied. The aeroplanes used were of the familiar steel tube, fabric-covered, tandem, two-seat type.—(5.3).

Appraisal of hazards to human survival in airplane crash fires. Gerard J. Pesman. N.A.C.A. T.N. 2996. (*September* 1953.)

The factors which affect the survival of human beings in aeroplane accidents followed by fire were studied by conducting full-scale crashes of transport and cargo-type aeroplanes. Studies of burning aeroplane hulks supplemented the information obtained from the crash fires. The time interval during which occupants could escape from a burning aeroplane was determined by using the time histories of cabin temperatures and toxic gas concentrations in conjunction with data that define the environmental conditions which can be tolerated by human beings. Other hazardous factors, such as flying detached aeroplane parts, explosions, and crushing of the aeroplane structure, were also studied.—(5.3).

Electrostatic spark ignition-source hazard in airplane crashes. A. M. Busch. N.A.C.A. Technical Note 3026 (October 1953).

The hazard of igniting aeroplane crash fires by electrostatic sparks, generated when detached aeroplane parts fly through clouds of dust and fuel mist, was investigated. Maximum rates of experimental electrification were used to relate energy accumulation to wreckage sizes and trajectories and to estimate minimum hazardous wreckage sizes and trajectories.—(5.3).

An analytical study of the effect of airplane wake on the lateral dispersion of aerial sprays. W. H. Reed, III. N.A.C.A. Technical Note 3032 (*October* 1953).

An analysis is made to determine the trajectories and deposit of aerial spray droplets which are issued into the air disturbances generated by an agricultural aeroplane. Various nozzle arrangements and droplet-size spectra are considered with a view to improving the uniformity and effective width of the deposit.—(5.4.1).

ELECTRONICS

The measurement of true air speed by a radar method. F. W. Hooton and R. W. Murphy. A.R.L. Australia. Flight Note 18 (*July* 1953).

Flight tests are described of a method of measuring true air speed, using ground radar equipment, which obviates the need for wind finding. The application of the scheme to the determination of static pressure error is given in some detail.—(11 × 1.12).

FUELS AND LUBRICANTS

Hydrocarbon and nonhydrocarbon derivatives of cyclopropane. V. A. Slabey, P. H. Wise and L. C. Gibbons. N.A.C.A. Report 1112 (1953).

The methods used to prepare and purify 19 hydrocarbon derivatives of cyclopropane are discussed. Of these hydrocarbons, 13 were synthesised for the first time. In addition to the hydrocarbons, six cyclopropylcarbinols, five alkyl cyclopropyl ketones, three cyclopropyl chlorides, and one cyclopropanedicarboxylate were prepared as synthesis intermediates. The melting points, boiling points, refractive indices, densities and, in some instances, heats of combustion of both the hydrocarbon and nonhydrocarbon derivatives of cyclopropane were determined.—(14.3).

HYDRODYNAMICS

Etude théorique et expérimentale de la propagation des intumescences dans les canaux découverts. J. Nougaro. Publications Scientifiques et Techniques du Ministère de L'Air, France, No. 284 (1953).—(17.1 × 1.2.3).

INSTRUMENTS AND EQUIPMENT

See also PROPELLERS

*Electro-thermal de-icing systems: their design and control. J. L. Orr, J. H. Milsum and C. K. Rush. N.A.E. Canada, Laboratory Report LR-*70 (*March* 1953).

If, instead of an "anti-icing" (or ice prevention) principle, a "de-icing" (or ice removal) principle is employed, a great reduction in total heat requirements can be obtained. In this case, a layer of ice is allowed to form and is then periodically shed by heating the surface to which it is attached. The resulting liquid interface reduces the adhesion, thereby permitting removal of the ice by aerodynamic or centrifugal forces. The electro-thermal de-icing principle was first successfully applied to propellers and is now coming into general use. This application is described and optimum power inputs, heat distribution and coverage are discussed.—(18.4).

*The characteristics of an orifice-type icing-detector probe. D. Fraser. N.A.E. Canada. Laboratory Report LR-*71 (*June* 1953).

A further investigation of the icing characteristics of an orifice-type icing-detector probe has been made in the icing tunnel. It was found that the probe was not completely effective in detecting the extreme forms of icing which occur either at high speeds and temperatures close to freezing, or at low temperatures and low rates of catch. It appears that any dangerous form of icing will probably be detected.—(18.4).

An empirically derived basis for calculating the area, rate, and distribution of water-drop impingement on airfoils. N. R. Bergrun. N.A.C.A. Report 1107 (1952).

An empirically derived basis for predicting the area, rate, and distribution of water-drop impingement on aerofoils of arbitrary section is presented. The concepts involved represent an initial step towards the development of a calculation technique which is generally applicable in the design of thermal ice-prevention equipment for aeroplane wing and tail surfaces.—(18.4).

Maximum evaporation rates of water droplets approaching obstacles in the atmosphere under icing conditions. H. H. Lowell. N.A.C.A. Technical Note 3024 (*October* 1953).

Maximum possible evaporation rates of water droplets approaching obstacles in the atmosphere along stagnation lines, or moving within intake ducts of aeroplanes under icing conditions were calculated for a wide variety of ambient conditions, flight Mach numbers, degrees of stagnation of the incident relative air stream, and droplet diameters. —(18.4).

MATERIALS

See also MATHEMATICS

Polymethyl methacrylate (*Perspex type*) *plastics; crazing, thermal and mechanical properties. H. Warburton Hall and E. W. Russell. R. & M.* 2764. (*October* 1949, *published* 1953.)

The more practical aspects of the results of a long-term investigation of the basic physical and chemical properties of polymethyl methacrylate (" Perspex " type) plastic are summarised. Thermal, elastic, crazing, solvent absorption and mechanical properties are included and the effect of these on the service efficiency of a plastic structure is described.—(21.3.3).

Les modifications de structure du cristal métallique et leur influence sur la cinétique du durcissement structural des solutions solides d'aluminium. A. Berghezan. Publications Scientifiques et Techniques du Ministère de L'Air, France. No. 283 (1953).—(21.2.2).

Coefficient of friction and damage to contact area during the early stages of fretting. I—Glass, copper, or steel against copper. Douglas Godfrey and John M. Bailey. N.A.C.A. T.N. 3011. (*September* 1953.)

Experiments were conducted to measure the coefficient of friction μ and to determine the damage to the contact area during early stages of fretting of copper at a frequency of 5 cycles per minute. Specimen combinations of copper against glass, copper against copper, and copper against steel, as well as various copper oxide films and powder compacts, were used.—(21.2.2)

Axial-load fatigue tests on notched and unnotched sheet specimens of 61S-T6 *aluminum alloy, annealed* 347 *stainless steel, and heat-treated* 403 *stainless steel. H. F. Hardrath, C. B. Landers and E. C. Utley, Jr. N.A.C.A. Technical Note* 3017 (*October* 1953).

Axial-load fatigue tests at a stress ratio of zero were performed on notched and unnotched sheet specimens of 61S-T6 aluminium alloy and 347 and 403 stainless steels. Special emphasis was placed on tests at high stress levels producing failures in small numbers of cycles.—(21.2.1 × 21.2.2).

Investigation of the statistical nature of the fatigue of metals. G. E. Dieter and R. F. Mehl. N.A.C.A. Technical Note 3019 (*September* 1953).

Results are presented of an investigation of the statistical nature of the fatigue of metals utilising statistical methods developed previously. The investigation included a study of the fatigue properties and their statistical variation of a plain carbon eutectoid steel heat-treated to coarse pearlitic and spheroidised structures of the same tensile strength and of commercially pure aluminium (2S) and 24S alloy heat-treated to two different structures. Calculation of data from the literature provided statistics of 75S aluminium alloy for comparison with the data of the present investigation.—(21.2.1 × 21.2.2).

Relative importance of various sources of defect-producing hydrogen introduced into steel during application of vitreous coatings. D. G. Moore, Mary A. Mason, and W. N. Harrison. N.A.C.A. Report 1120 (1953).

The purpose of this investigation was to study and to determine experimentally the relative importance of various sources of defect-producing hydrogen introduced into steel during the application of vitreous-type ceramic coatings. Under the conditions investigated, the principal source of defect-producing hydrogen was the dissolved water present in the enamel frit of the coating. Acid pickling, milling water, chemically combined water in the clay, and quenching water were all minor sources. Formerly T.N. 2617.—(21.3.1 × 21.4).

MATHEMATICS

L'algorithme de Gauss modernise et son application a des systèmes d'équations linéaires dégénérés ou mal ordonnés. R. Kappus. O.N.E.R.A., France. Note Technique No. 11. (*July* 1952, *published* 1953.)—(22.1).

*Note on the general stress-strain relations of some ideal bodies showing the phenomena of creep and of relaxation. J. P. Benthem. N.L.L. Holland. Report S.*426.

The general stress-strain relations of the Kelvin-Solid and the Maxwell-Liquid are derived from the cases of pure change of shape and of pure change of volume. It is shown that the relations of the Kelvin-Solid in pure tension may be much more complicated. The extension of the Kelvin-Solid and Maxwell-Liquid to general Linear Bodies is indicated. Some remarks are made about viscosity effects, which may accompany plastic deformation.—(22.3 × 21).

*A method of analyzing dynamic test data. R. Sedney and D. Zes. Douglas Report No. SM-*14774. (*May* 1953.)

The problem of obtaining the constants of a physical system from dynamic test data is considered. It is assumed that the test data contain random errors. The constants solved for are coefficients in the equations of motion of the system. The averaging procedure developed requires that the constants enter the equations of motion linearly; the equations may contain non-linear functions of the co-ordinates. A criterion for the number of constants that may be obtained is discussed. Some conclusions based on numerical experiments and the results of one case are given. —(22.2).

MECHANICAL ENGINEERING

Comparison of operating characteristics of four experimental and two conventional 75-millimeter-bore cylindrical-roller bearings at high speeds. William J. Anderson, E. Fred Macks and Zolton N. Nemeth. N.A.C.A. T.N. 3001. (*September* 1953.)

Studies of four experimental outer race-riding cage-type bearings with inner race-guided rollers, a conventional inner race-riding cage-type bearing, and a conventional outer race-riding cage-type bearing with outer race-guided rollers are reported. All cages were made of nodular iron.—(23.2.1).

Effect of bronze and nodular iron cage materials on cage slip and other performance characteristics of 75-millimeter-bore cylindrical-roller bearings at DN values to 2×10^5. *William J. Anderson, E. Fred Macks and Zolton N. Nemeth. N.A.C.A. T.N.* 3002. (*September* 1953.)

In an investigation of four outer race-riding cage-type bearings (two with bronze and two with nodular iron cages) heavy wear was found to accompany cage slip. Nodular iron seemed to promote cage slip at DN values in excess of $1{\cdot}2 \times 10^6$; consequently, bearings with bronze cages showed less wear than did bearings with nodular iron cages at very high speeds. There was little difference in performance between the bronze and nodular iron at DN values to $1{\cdot}2 \times 10^5$.—(23.2.1).

Investigation of 75-millimeter-bore deep-groove ball bearings under radial load at high speeds. II—Oil inlet temperature, viscosity, and generalized cooling correlation. Zolton N. Nemeth, E. Fred Macks and William J. Anderson. N.A.C.A. T.N. 3003. (*September* 1953.)

The extent of the effects of oil inlet temperature and viscosity on bearing inner and outer-race temperatures is shown. In this investigation, 75-millimeter-bore ball bearings were operated at DN values (bore times speed) from $0{\cdot}3 \times 10^6$ to $2{\cdot}4 \times 10^6$, radial loads from 7 to 1113 lb., and oil flows from 1·6 to 8 lb./min. The cooling correlation previously developed for cylindrical-roller-bearing temperatures was extended and applied to ball bearing temperatures and to the power rejected to the oil by the test bearing.—(23.2.1).

POWER PLANTS

See also RESEARCH

An analysis of turbojet-engine-inlet matching. D. D. Wyatt. N.A.C.A. Technical Note 3012 (*September* 1953).

A method of presenting turbojet-engine- air flow requirements and inlet-system air flow capacities in identical, though

independent, parametric terms is developed. The application of the air flow representation technique to the analysis of engine-inlet matching conditions is demonstrated. Several examples are presented to illustrate the application of the method to the explicit determination of inlet geometric variations required to improve the power-plant performance of supersonic aeroplanes.—(27.1).

Influence of rotor-engine torsional oscillation on control of gas-turbine engine geared to helicopter rotor. J. C. Sanders. N.A.C.A Technical Note 3027 (*October* 1953).

Equations were developed for the torsional motion of a gas-turbine engine geared to a helicopter rotor in which the rotor blades were hinged to the rotor shaft. The rotor system was simplified to yield simple third-order equations that can be used in the analysis of engine control.—(27.1).

PROPELLERS

See also THERMODYNAMICS *and* INSTRUMENTS AND EQUIPMENT

Linearized potential theory of propeller induction in a compressible flow. Robert E. Davidson. N.A.C.A. T.N. 2983. (*September* 1953.)

The potential-theory representation of the wave-equation flow about a lifting-line propeller of finite number of blades and arbitrary circulation distribution is given. From the velocity potential, the compressible inflow velocities at the blade became known. The induced velocities are known also at any point in the flow because the velocity potential is determined for the whole field.—(29.1).

A theoretical study of the effect of forward speed on the free-space sound-pressure field around propellers. I. E. Garrick and C. E. Watkins. N.A.C.A. Technical Note 3018 (*October* 1953).

The sound-pressure field due to thrust and torque of a propeller in flight at uniform subsonic speed is analysed by use of a distribution of acoustic doublets located at the propeller disc. The results can be regarded as an extension of the work for the static propeller given by Gutin (N.A.C.A. T.M. 1195) for the far pressure field and given by Hubbard and Regier (N.A.C.A. Rep. 996) for the field near the tips of the rotating propeller.—(29.1).

An investigation utilizing an electrical analogue of cyclic de-icing of hollow steel propellers with internal electric heaters. C. B. Neel, Jr. N.A.C.A. Technical Note 3025 (*October* 1953).

A study has been made of the heating requirements for the cyclic de-icing of hollow steel propellers fitted with two tyes of internal electric heaters. Solutions to the transient-heat-flow equations depicting the cyclic de-icing of propellers were obtained by use of an electrical analogy.—(29.3 × 18.4).

RESEARCH

Application of an electro-optical two-color pyrometer to measurement of flame temperature for liquid oxygen-hydrocarbon propellant combination. M. F. Heidmann and R. J. Priem. N.A.C.A. Technical Note 3033 (*October* 1953).

The development of an electro-optical two-colour pyrometer and its application to the measurement of temperature and emissivity in the exhaust gases of an open-tube combustor during normal and oscillatory combustion are described. The results of simultaneous measurements of sound intensity and temperature by the two-colour method and by a microwave absorption technique are also presented.—(31.3 × 27.3).

STRUCTURES

LOADS

See also AEROELASTICITY

Interim report on V-g records on helicopters. H. I. Birds. R. & M. 2746 (*March* 1949, *published* 1953).

V-g records have been obtained on Hoverfly I helicopers and also some data on a Hoverfly II and a Sikorsky S.51. The *V-g* records on these aircraft were obtained mainly during test flying, which included blind flying and some general flying.—(33.1.2).

*An approach to the buffeting of aircraft structures by jets. J. W. Miles. Douglas Report No. SM-*14795. (*June* 1953.)

Experience has shown that the fluctuating loads induced by a jet may cause fatigue failure of aircraft structural components. To throw some light on this and similar problems, the stress spectrum and the "equivalent fatigue stress" of an elastic structure subjected to random loading are studied. As an example, a similarity expression for the probable value of the equivalent fatigue stress of a panel subjected to jet buffeting is derived.—(33.1.2).

Correlation of calculation and flight studies of the effect of wing flexibility on structural response due to gusts. John C. Houbolt. N.A.C.A. T.N. 3006. (*August* 1953.)

Flight and calculation studies of two twin-engine transports and one four-engine bomber are presented to show how wing bending flexibility influences the structural response due to gusts. It is shown that an analysis approach based upon single-gust encounter reveals the general nature of the flexibility effects and leads to qualitative correlation with flight results. It is shown further that quantitative correlation can be obtained with an approach that considers continuous-turbulence encounter and is based on generalised harmonic analysis.—(33.1.2 × 2).

THEORY AND ANALYSIS

See also THERMODYNAMICS

The buckling in compression of panels with square top-hat section stringers. W. S. Hemp and K. H. Griffin. R. & M. 2635 (*June* 1949, *published* 1953).

A simplified panel model is described, together with a number of assumptions about the mode of its buckling. The approach to the calculation of the buckling stress is by splitting the panel into a number of flat plates and treating these by the ordinary plate theory. Use of the boundary conditions betwen these plates leads to a relation between the buckling stress and the variables of the panel geometry. The results thus obtained are compared with two sets of experimental work; an appendix is included to show the effect of initial panel irregularities on the experimental determination of buckling stresses.—(33.2.4).

Investigation of skin buckling. D. J. Farrar. R. & M. 2652 (*October* 1947, *published* 1953).

The present tests were conducted on aluminium alloy plates in endwise compression, with varying conditions of edge support, to provide data on the buckling stress and post-buckling behaviour of aircraft skins. Measurements were made of the plate load and mean strain, and the shape of the skin buckles. The test technique is discussed and the experimental results compared with theory. This report on an Investigation of Skin Buckling was originally prepared by the Bristol Aeroplane Co. Ltd., as Report No. C.R. 434 in October 1945. The test results are given in detail in S. & T.M. 12/47.—(33.2.4).

The initial buckling of a long and slightly bowed panel under combined shear and normal pressure. E. H. Brown and H. G. Hopkins. R. & M. 2766 (*June* 1949, *published* 1953).

Recent American experimental work has suggested that the resistance to buckling of wing skin panels under compression or shear loads is improved by aerodynamic suction. A complete theoretical analysis of this problem is difficult, because compression load necessarily involves the consideration of post-buckling behaviour. An approach is made in this report by considering the restricted problem of the initial buckling of a long, thin and slightly bowed panel under combined shear and normal pressure.—(33.2.4).

*Buckling of oblique plates with clamped edges under uniform shear. W. H. Wittrick. A.R.L. Australia. Report SM.*210. (*June* 1953.)

In a previous paper (A.R.L. Report No. SM.182), the author derived (by the Rayleigh-Ritz method) a determinantal equation for obtaining the critical magnitude of any uniform system of edge stress applied to a clamped oblique plate. In the present paper this equation is used to derive values for the buckling stresses of clamped oblique plates subjected to pure shear along and perpendicular to two edges of the plate.—(33.2.1).

APPOINTMENTS

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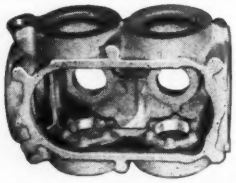

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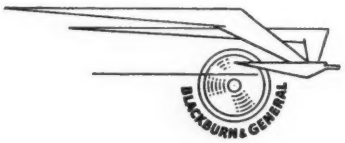

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Saunders Valve Co. Ltd.	Cwmbran, Newport, Monmouthshire	Cwmbran 3281-3
Self-Priming Pump & Engineering Co. Ltd.	Edinburgh Avenue, Trading Estate, Slough, Bucks.	Slough 23277
Serck Radiators Ltd.	Warwick Road, Birmingham, 11	Victoria 0531
Shell-Mex and B.P. Ltd.	Shell-Mex House, Strand, London, W.C.2	Temple Bar 1234
Short Brothers and Harland Ltd.	Seaplane Works, Queen's Island, Belfast	Belfast 58444
Smiths Aircraft Instruments Ltd.	Cricklewood Works, London, N.W.2	Gladstone 3333
Sperry Gyroscope Co. Ltd., The	Great West Road, Brentford, Middlesex	Ealing 6771
Standard Telephones and Cables Ltd.	Connaught House, Aldwych, London W.C.2	Holborn 8765
Tungum Co. Ltd.	Brandon House, Painswick Road, Cheltenham, Glos.	Cheltenham 5856
Vickers - Armstrongs Ltd.	Vickers House, Broadway, Westminster, S.W.1	Abbey 7777
	Weybridge Works, Weybridge, Surrey	Byfleet 240-243
	Supermarine Works, Hursley Park, Winchester, Hants.	Chandlersford 2251
Vokes Ltd.	Henley Park, nr. Guildford, Surrey	Guildford 62861
Wakefield, C. C., & Co. Ltd.	46 Grosvenor Street, London, W.1	Mayfair 9232
Westland Aircraft Ltd.	Yeovil, Somerset	Yeovil 1100

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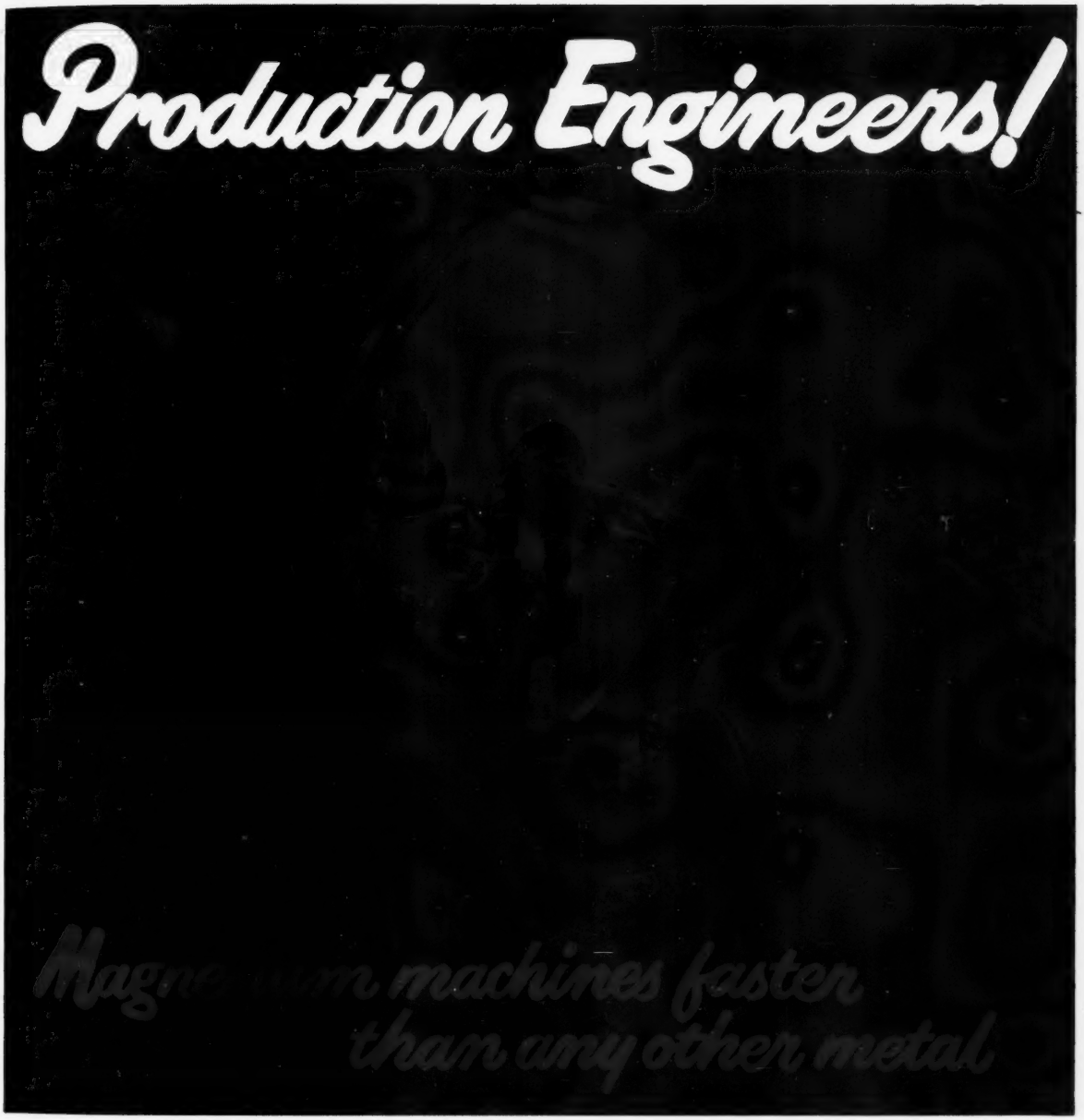

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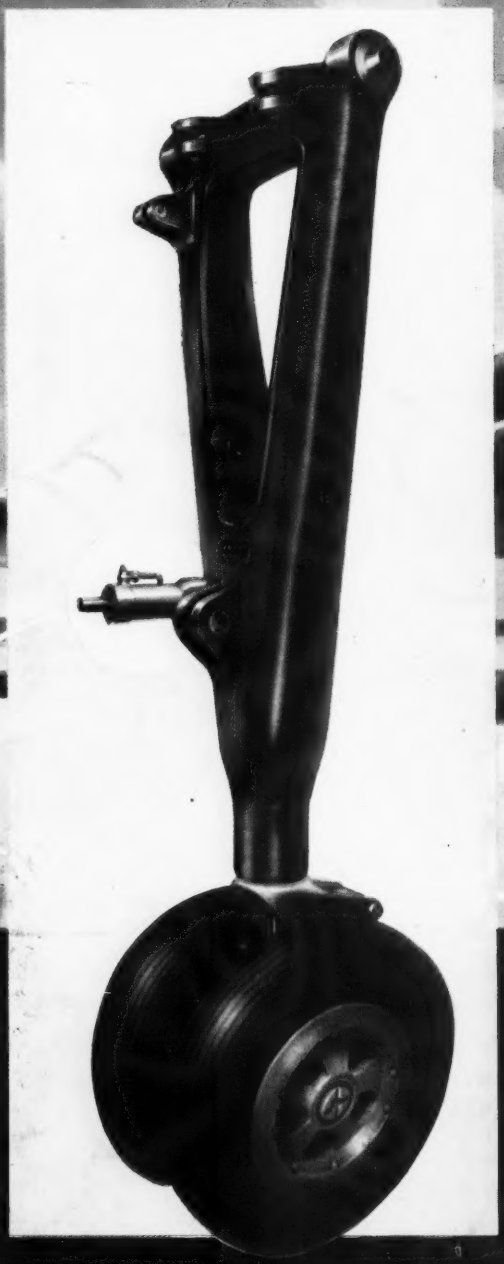

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